# An argument for the second-derivative test

We turn our attention now to developing a test to distinguish among local minima, local maxima, and saddle points for critical inputs. The test involves the second partial derivatives of the function $f : \mathbb{R}^2 \to \mathbb{R}$. We assume that $f$ has continuous second partial derivatives, so that the mixed partials are equal.

Let $(x_0, y_0)$ be a critical input for $f$. By definition, the first partial derivatives of $f$ are zero for input $(x_0, y_0)$. Using this fact, the second-order Taylor polynomial approximation of $f$ based at $(x_0, y_0)$ gives

$$f(x,y) - f(x_0,y_0) \approx \frac{1}{2}\left[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\right]. \quad (1)$$

We want to determine the conditions under which the expression on the right side is

- positive for all $(x, y)$ near $(x_0, y_0)$, in which case $(x_0, y_0)$ is a local maximizer;
- negative for all $(x, y)$ near $(x_0, y_0)$, in which case $(x_0, y_0)$ is a local minimizer; or
- positive for some $(x, y)$ near $(x_0, y_0)$ and negative for other $(x, y)$ near $(x_0, y_0)$, in which case $(x_0, y_0)$ is a local "saddlizer."

To simplify the expression on the right side of Equation (1), we introduce the notation

$$A = f_{xx}(x_0, y_0), \qquad B = f_{xy}(x_0, y_0)\text{ , and} \qquad C = f_{yy}(x_0, y_0)$$

These are constants for a given function and a given critical input. We also let

$$u = x - x_0 \qquad \text{and} \qquad v = y - y_0.$$

We focus on $(u, v) = (0, 0)$, because this input in $u$ and $v$ corresponds to the critical input $(x, y) = (x_0, y_0)$. We drop the factor of $1/2$ because this will not affect the sign of the expression. With this notation, the expression on the right side in Equation (1) gives the outputs of a function

$$g(u, v) = Au^2 + 2Buv + Cv^2.$$

This is a quadratic function in the input variables $u$ and $v$.

To understand the shape of the graph of $g$ near $(0, 0)$, we examine slices with vertical planes containing the $z = g(u, v)$ axis. These planes are defined by lines throught the origin in the $uv$-plane. Along the line given by $v = mu$, the outputs of $g$ are

$$g(u, mu) = Au^2 + 2Bu(mu) + C(mu)^2 = (A + 2Bm + Cm^2)u^2.$$

From this last expression, we deduce that the concavity of the curve in the section defined by $v = mu$ is determined by the sign of

$$A + 2Bm + Cm^2.$$

The sign clearly depends on the value of $m$, that is, on which section we have. If the concavity is positive for every value of $m$, then the critical input at $(u, v) = (0, 0)$ must be a local minimizer. If the concavity is negative for every value of $m$, then the critical input must be a local maximizer. If the concavity is negative in some sections and positive in others, then the critical input corresponds to a saddle.

Note that the line $u = 0$ is not included among the lines $v = mu$. The line $u = 0$ is the $v$ cross section. The concavity in this section is given by $C$. Note this is consistent with the fact that the $v = 0$ section has concavity given by $A$.

To distinguish among the possible cases, we look at the condition under which the concavity is zero. Let $m_0$ be a value for which

$$A + 2Bm_0 + Cm_0^2 = 0.$$

The case $C = 0$ is a special situation, which you should analyze on your own. Assuming that $C$ is not equal to zero, we use the quadratic formula to solve for $m_0$ giving

$$m_0 = \frac{-2B \pm \sqrt{4B^2 - 4AC}}{2C} = \frac{-B \pm \sqrt{B^2 - AC}}{C}. \tag{2}$$

There are two possibilities, depending on the sign of the quantity $B^2 - AC$. If $B^2 - AC > 0$, then there are two solutions for $m_0$, corresponding to two sections in which the concavity is zero. These lines divide the $uv$-plane into four "wedges," and the concavity alternates from positive to negative in these wedges (see Figure 1). The critical input $(0, 0)$ thus corresponds to a saddle. If $B^2 - AC < 0$, then there are no (real-valued) solutions. Hence, the concavity is positive in all sections or negative in all sections. Note that in order for this case to occur, $A$ and $C$ must have the same sign. If the signs are both positive, then all sections are concave up, meaning $(0, 0)$ is a local minimizer. If the signs are both negative, then all sections are concave down, meaning $(0, 0)$ is a local maximizer.

We summarize the conclusions of the preceding argument in the following theorem. The argument above constitutes a proof of the theorem. For historical reasons, the convention is to define the quantity $AC - B^2$ as the discriminant. We denote this as $D$.

**Theorem 1.** *Suppose* $(x_0, y_0)$ *is a critical input for a function* $f : \mathbb{R}^2 \to \mathbb{R}$. *Also, suppose* $f$ *has continuous second partial derivatives in some open disk centered at* $(x_0, y_0)$. *Let*

$$D = AC - B^2 = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \left[f_{xy}(x_0, y_0)\right]^2.$$

*If* $D < 0$, *then* $(x_0, y_0)$ *corresponds to a saddle point. If* $D > 0$, *then* $(x_0, y_0)$ *is a local extremizer. In this case, if* $A > 0$ *and* $C > 0$ *then* $(x_0, y_0)$ *is a local minimizer. If* $A < 0$ *and* $C < 0$, *then* $(x_0, y_0)$ *is a local maximizer.*

Note that the theorem is silent in the case $D = AC - B^2 = 0$.

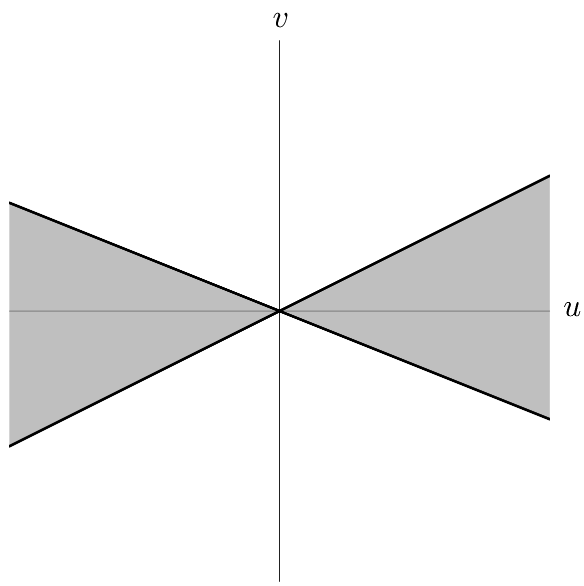


Figure 1: For a quadratic saddle there are two sections in which the concavity is zero. These correspond to the two lines through the origin in the $uv$-plane. The concavity is the same sign for all sections in the gray "wedges" and of opposite sign for all sections in the white "wedges."