Finding potential functions

Version 2 (April 22, 2004)

A vector field $\vec{F}$ has a potential function V if $\vec{\nabla} V = \vec{F}$. If the vector field is planar, $F : \mathbb{R}^2 \to \mathbb{R}^2$, then the potential function must be a function of two variables, $V : \mathbb{R}^2 \to \mathbb{R}$. The general problem is to determine if a given vector field $\vec{F}$ has a potential function and, if so, find a potential function.

Determining if a planar vector field has a potential function

Let's start by assuming that the planar vector field $\vec{F}$ does have a potential function V. Then we know $\vec{\nabla} V = \vec{F}$. Introducing coordinates, we can write this out as

$$\frac{\partial V}{\partial x}\,\hat{\imath} + \frac{\partial V}{\partial y}\,\hat{\jmath} = u\,\hat{\imath} + v\,\hat{\jmath}.$$

Equating components gives us the two scalar equations

$$\frac{\partial V}{\partial x} = u \qquad \text{and} \qquad \frac{\partial V}{\partial x} = v.$$

Now compute the partial with respect to y on both sides of the first equation and the partial with respect to x on both sides of the second equation to get

$$\frac{\partial^2 V}{\partial y \partial x} = \frac{\partial u}{\partial y} \qquad \text{and} \qquad \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial v}{\partial x}.$$

By equality of mixed partials, we have

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

(Note that we need to assume V satisfies the conditions of the theorem that gives equality of mixed partial derivatives here. You can go back and look these up.) We thus have proven the statement

If $\vec{F} = u\,\hat{\imath} + v\,\hat{\jmath}$ has a potential function, then $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$.

This is a statement of the form *If A, then B.* For any "if-then" implication, the *contrapositive* statement *If not B, then not A* is logically equivalent. Thus we have proven the statement

If $\dfrac{\partial u}{\partial y} \neq \dfrac{\partial v}{\partial x}$, then $\vec{F} = u\,\hat{\imath} + v\,\hat{\jmath}$ does not have a potential function.

The statement and its contrapositive are half of Theorem 13.3 (the "cross-partials test") in the text. The other half of the theorem is the statement

If $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$, then $\vec{F} = u\,\hat{\imath} + v\,\hat{\jmath}$ has a potential function.

We have not proven this statement.

To be complete, these statements should include another hypothesis. This is the requirement that *the component functions u and v have continuous first partial deriviatives in an open, simply connected region of the plane.* You can read about the terms *open* and *simply connected* in the text.

Finding a potential function

Problem: Find a potential function for the vector field $\vec{F}(x, y) = x\,\hat{\imath} + y\,\hat{\jmath}$.
Solution: First, let's use the "cross-partials test" to check if $\vec{F}$ has a potential function. The component functions $u(x, y) = x$ and $v(x, y) = y$ clearly have continuous first partial derivatives for the entire plane. Now compute and find

$$\frac{\partial u}{\partial y} = 0 = \frac{\partial v}{\partial x}$$

for all (x, y). Thus $\vec{F}$ has a potential function for all (x, y).

Let $V(x, y)$ be the potential function for $\vec{F}$. By definition

$$\frac{\partial V}{\partial x} = u(x, y) = x \qquad \text{and} \qquad \frac{\partial V}{\partial y} = v(x, y) = y.$$

We can (indefinitely) integrate each of these with respect to the appropriate variable to get

$$V(x, y) = \frac{1}{2}x^2 + \phi(y) \qquad \text{and} \qquad V(x, y) = \frac{1}{2}y^2 + \psi(x).$$

Note that because we are holding y constant in the first integration, the "constant of integration" can be a function of y. Another way to think about this is to to take the partial derivative of the first expression for V with respect to x and see that we recover what we started with since the partial derivative of $\phi(y)$ with respect to x is 0. In a similar fashion, the "constant of integration" in the second expression can be a function of x.

Now, there is only one potential function $V(x, y)$, so we must pick $\phi(y)$ and $\psi(x)$ to make the two expressions above consistent. Here, it is easy to see that the choice

$$\phi(y) = \frac{1}{2}y^2 \qquad \text{and} \qquad \psi(x) = \frac{1}{2}x^2$$

works. The potential function is thus

$$V(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 = \frac{1}{2}(x^2 + y^2).$$

Problem: Show that the vector field $\vec{F}(x, y) = -y\,\hat{\imath} + x\,\hat{\jmath}$ does not have a potential function.
Solution: We use the "cross-partials test" to check if $\vec{F}$ has a potential function. The component functions $u(x, y) = -y$ and $v(x, y) = x$ clearly have continuous first partial for the entire plane. Now compute and find

$$\frac{\partial u}{\partial y} = -1 \qquad \text{and} \qquad \frac{\partial v}{\partial x} = 1$$

for all (x, y). Thus there is no region in the plane on which $\vec{F}$ has a potential function.

Let's go a little bit further here by exploring what happens if we look for this nonexistent potential function. Let $V(x, y)$ be a potential potential function (this is not a typo) for $\vec{F}$. By definition

$$\frac{\partial V}{\partial x} = u(x, y) = -y \qquad \text{and} \qquad \frac{\partial V}{\partial y} = v(x, y) = x.$$

We can (indefinitely) integrate each of these with respect to the appropriate variable to get

$$V(x, y) = -xy + \phi(y) \qquad \text{and} \qquad V(x, y) = xy + \psi(x).$$

The key here is that *no* choice of $\phi(y)$ and $\psi(x)$ will make these consistent.

Thinking geometrically about the existence of a potential function

Suppose we have a planar vector field $\vec{F}$. We want to think about whether $\vec{F}$ has a potential function or not. By definition, the function V is a potential function for $\vec{F}$ if $\vec{\nabla} V = \vec{F}$. Recall that at a point $P(x, y)$, the gradient vector $\vec{\nabla} V(x, y)$ is perpendicular to the level curve of V that goes through P. If we have a plot of the given vector field $\vec{F}$, we can start by drawing, at the base of each vector, a short line segment perpendicular to that vector. The question we have to ask is *Can we connect these line segments to form level curves for the potential function V?* The level curves of a function can not intersect. (You should think through why this is so.)

To test out this idea, draw a vector field plot for each of the vector fields in the two problems above. On each vector field plot, draw the perpendicular line segments and see if you can connect these up to form sensible level curves.

The *Vector Field Analyzer* provides some tools to help with this geometric view of potential functions. The first is right above the "Plot this field" button. You will see the phrases "Arrows (contra-var)" and "Stacks (co-var)." The default is "Arrows". Click on the button closest to "Stacks" and look for the change in the plot window. Each arrow is replaced by a "stack" of line segments perpendicular to the corresponding arrow. The density of the stack is proportional to the length of the corresponding arrow. To make this a little more obvious, you might want to change the value in the box labeled "grid" just under the second component window near the bottom. The default value is 20 meaning that arrows/stacks are plotted on a 20 by 20 grid for a total of 400 arrows/stack. If you change this value to 10 (and then hit the "Plot this field" button), you get a 10 by 10 grid. With fewer arrows/stacks, each can be bigger without overlap.

These stacks represent pieces of potential level curves. Can these be joined up into sensible level curves? Try the vector field $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$. Compare what you see with the results of the first problem above. Try the vector field $\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath}$. Compare what you see with the results of the second problem above.

The second tool is under the "DEs/flows" tab. On this tab, click on the button labeled "Equipot. candidates." Then, click on one or more points inside the plot window. A blue dot will be drawn at each point on which click in the window. Finally, click on the button labeled "Stop and Go." For each point you made, the program will start drawing a curve that is perpendicular to the vector field arrows (and parallel to the stacks if you are in that view). Experiment here with the same two vector fields used above.

If you have corrections or suggestions for improvements to these notes, please contact Martin Jackson, Department of Mathematics and Computer Science, University of Puget Sound, Tacoma, WA 98416, martinj@ups.edu.