# Circulation density and an argument for Stokes' Theorem

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There are three sections in these notes. The goal for the first section is to see that the curl of a vector field can be thought of in terms of something we will call *circulation density*. In the second section, we look at adding circulation for adjacent rectangles. In the third section, we will outline a proof of Stokes' Theorem. This proof uses the ideas from the first two sections.

## Circulation density and curl

Given a vector field $\vec{F}$ and an oriented closed loop $C$ in space, we can think of the line integral $\oint_C \vec{F} \cdot d\vec{R}$ as a *circulation*. That is, interpret $\vec{F}$ as velocity of a fluid flow and think about the curve $C$ as a rigid wire in this flow. Circulation is a measure of how much the flow helps if we push a bead around the wire in the direction specified by the orientation.

Now imagine a small flat loop at the point $P$. We will use $\Delta C$ to denote this small flat loop. Here $\Delta$ doesn't mean "a small change in" but serves to remind us that the loop is small. For simplicity, we can think of this small flat loop as a rectangle. Let $\Delta A$ be the area enclosed by the loop. Both the circulation $\oint_{\Delta C} \vec{F} \cdot d\vec{R}$ and the area $\Delta A$ will go to zero as we shrink the loop $\Delta C$ down to the point $P$. However, the limit of the ratio

$$\frac{\displaystyle\oint_{\Delta C} \vec{F} \cdot d\vec{R}}{\Delta A}$$

might exist. This limit is the *circulation density*. That is, the limit of the ratio is the circulation per unit area. The goal of this section is to see how the circulation density is related to the curl $\vec{\nabla} \times \vec{F}$ of the vector field.

We will achieve this goal by looking in detail at a special case. Let $\Delta C$ be a small rectangle parallel to the $xy$-plane oriented counter-clockwise as viewed from above. (See Figure 1.) Let $P$ be one corner of this rectangle. Let $\Delta x$ and $\Delta y$ be the side lengths parallel to the $x$-axis and $y$-axis respectively. The area of the rectangle is $\Delta A = \Delta x \, \Delta y$. In Figure 1, the area vector $\Delta\vec{A}$ is shown in red. This area vector can be written as $\Delta\vec{A} = \Delta A \, \hat{k}$ in this case. (You should think about why this is so.)

Now let's think about the circulation for this rectangle as given by the line integral $\oint_{\Delta C} \vec{F} \cdot d\vec{R}$. Since we will be eventually taking a limit as the rectangle shrinks to the point $P$, we can approximate this line integral with just four terms, one for each side of the rectangle. Let $\Delta\vec{R}_1$, $\Delta\vec{R}_2$, $\Delta\vec{R}_3$, and $\Delta\vec{R}_4$ be the displacement vectors along the four sides as shown in Figure 2. For each side, we can choose where to evaluate the vector field $\vec{F}$. We will use the corners $P$, $P_1$, and $P_2$ shown in Figure 2. Specifically, we use

$$\oint_{\Delta C} \vec{F} \cdot d\vec{R} \approx \vec{F}(P) \cdot \Delta\vec{R}_1 + \vec{F}(P_1) \cdot \Delta\vec{R}_2 + \vec{F}(P_2) \cdot \Delta\vec{R}_3 + \vec{F}(P) \cdot \Delta\vec{R}_4. \qquad (1)$$

Now let's introduce coordinates and components for the vectors in Equation (1). Let the vector field $\vec{F}$ have components

$$\vec{F} = u\,\hat{\imath} + v\,\hat{\jmath} + w\,\hat{k}.$$

We choose coordinates with $P(x, y, z)$. Since the side lengths are $\Delta x$ and $\Delta y$, this gives us coordinates $P_1(x + \Delta x, y, z)$ and $P_2(x, y + \Delta y, z)$. From the geometry, we can see that $\Delta\vec{R}_1 = \Delta x\,\hat{\imath}$. Thus

$$\vec{F}(P) \cdot \Delta\vec{R}_1 = \Big(u(x,y,z)\,\hat{\imath} + v(x,y,z)\,\hat{\jmath} + w(x,y,z)\,\hat{k}\Big) \cdot \big(\Delta x\,\hat{\imath}\big) = u(x,y,z)\Delta x.$$

The other three terms are similar:

$$\begin{aligned}
\Delta\vec{R}_2 &= \quad \Delta y\,\hat{\jmath} \quad \text{so} \quad \vec{F}(P_1) \cdot \Delta\vec{R}_2 = v(x, y + \Delta x, z)\Delta y \\
\Delta\vec{R}_3 &= -\Delta x\,\hat{\imath} \quad \text{so} \quad \vec{F}(P_2) \cdot \Delta\vec{R}_3 = -u(x, y + \Delta y, z)\Delta x \\
\Delta\vec{R}_4 &= -\Delta y\,\hat{\jmath} \quad \text{so} \quad \vec{F}(P) \cdot \Delta\vec{R}_4 = -v(x, y, z)\Delta y
\end{aligned}$$

Stop here and make sure you understand why there are negative signs in the last two of these.

Now substitute into Equation (1) to get

$$\begin{aligned}
\oint_{\Delta C} \vec{F} \cdot d\vec{R} &\approx u(x,y,z)\Delta x + v(x,y+\Delta x,z)\Delta y - u(x,y+\Delta y,z)\Delta x - v(x,y,z)\Delta y \\
&= \big[u(x,y,z) - u(x,y+\Delta y,z)\big]\Delta x + \big[v(x,y+\Delta x,z) - v(x,y,z)\big]\Delta y \\
&= -\big[u(x,y+\Delta y,z) - u(x,y,z)\big]\Delta x + \big[v(x,y+\Delta x,z) - v(x,y,z)\big]\Delta y
\end{aligned}$$

Thus, the ratio of circulation to area enclosed is

$$\begin{aligned}
\frac{\oint_{\Delta C} \vec{F} \cdot d\vec{R}}{\Delta A} &= \frac{-\big[u(x,y+\Delta y,z) - u(x,y,z)\big]\Delta x + \big[v(x,y+\Delta x,z) - v(x,y,z)\big]\Delta y}{\Delta x\,\Delta y} \\
&= -\frac{u(x,y+\Delta y,z) - u(x,y,z)}{\Delta y} + \frac{v(x,y+\Delta x,z) - v(x,y,z)}{\Delta x}. \qquad (2)
\end{aligned}$$

Now consider the limit as $\Delta C$ shrinks to the point $P$. We achieve this by taking $\Delta x \to 0$ and $\Delta y \to 0$. Note each of the two terms on the right side of Equation (2) will give a partial derivative in this limit. We thus have

$$\text{circulation density at } P \;=\; \lim_{\Delta x, \Delta y \to 0} \frac{\oint_{\Delta C} \vec{F} \cdot d\vec{R}}{\Delta A} = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad (3)$$

for an "infinitesimal loop" parallel to the $xy$-plane. Another way to specify this orientation is to say the area vector for this "infinitesimal loop" is $d\vec{A} = dA\,\hat{k}$.

Now recall that the curl $\vec{\nabla} \times \vec{F}$ is given in components by

$$\vec{\nabla} \times \vec{F} = \Big(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\Big)\,\hat{\imath} - \Big(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\Big)\,\hat{\jmath} + \Big(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\Big)\,\hat{k}.$$

Compare the $\hat{k}$-component of this with the last expression in Equation (3). These are the same so we conclude

*The circulation density at a point $P$ for a loop with $d\vec{A} = dA\,\hat{k}$ is equal to the $\hat{k}$-component of $\vec{\nabla} \times \vec{F}$ at $P$.*

We can express the $\hat{k}$-component of $\vec{\nabla} \times \vec{F}$ as $(\vec{\nabla} \times \vec{F}) \cdot \hat{k}$ and write this conclusion as

$$\lim_{\Delta x, \Delta y \to 0} \frac{\oint_{\Delta C} \vec{F} \cdot d\vec{R}}{\Delta A} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k} \tag{4}$$

Now consider the more general situation in which $\Delta C$ is any small flat curve at a point $P$. (See Figure 3.) Let the corresponding 'infinitesimal loop" have an area vector $d\vec{A} = dA\,\hat{n}$ where $\hat{n}$ can point in any direction. From the special case (special in that we looked only at a rectangle parallel to the $xy$-plane), we generalize to the following claim:

*The circulation density at a point $P$ for a loop with $d\vec{A} = dA\,\hat{n}$ is equal to the $\hat{n}$-component of $\vec{\nabla} \times \vec{F}$ at $P$. That is,*

$$\lim_{\Delta A \to 0} \frac{\oint_{\Delta C} \vec{F} \cdot d\vec{R}}{\Delta A} = (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \tag{5}$$

From this result, we get

$$\oint_{\Delta C} \vec{F} \cdot d\vec{R} \approx (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \Delta A = (\vec{\nabla} \times \vec{F}) \cdot \Delta\vec{A} \tag{6}$$

for a small loop. That is, for a small loop, the circulation is approximately $(\vec{\nabla} \times \vec{F}) \cdot \Delta\vec{A}$. We will this use in the argument for Stokes' Theorem in the third section. Before doing so, we look at adding circulation for loops that share a common edge.

## Adding circulations

Consider two rectangular loops that share a common edge as shown in Figure 4. Let $C_1$ be the blue loop and $C_2$ be the red loop with orientations as shown. Let $C_3$ be the loop that consists of going around the outside of the large rectangle formed by removing the common edge. We now claim that

$$\oint_{C1} \vec{F} \cdot d\vec{R} + \oint_{C_2} \vec{F} \cdot d\vec{R} = \oint_{C_3} \vec{F} \cdot d\vec{R} \tag{7}$$

for this situation. To see this is true, think of breaking each of these rectangular curves into four pieces. By the properties of line integrals, we can express $\oint_{C_1} \vec{F} \cdot d\vec{R}$ as a sum of four line integrals, one over each side of the first rectangle. Likewise, we can express $\oint_{C_2} \vec{F} \cdot d\vec{R}$ as a sum of four line integrals, one over each side of the second rectangle. The sum $\oint_{C1} \vec{F} \cdot d\vec{R} + \oint_{C_2} \vec{F} \cdot d\vec{R}$ will have a total of eight terms. Two of these will involve the common edge. Since this is traversed once in each direction, these terms cancel. The remaining six terms can be put together to give $\oint_{C_3} \vec{F} \cdot d\vec{R}$.

This result is easily generalized to any polygonal loops that share a common edge.

**Proving Stokes' Theorem**

First, let's recall the statement of Stokes' Theorem: Let $\vec{F}$ be a vector field in space. Let $S$ be a surface in the domain of $\vec{F}$ with area vectors $d\vec{A}$ all on the same side. Let $C$ be the curve in space that forms the edge of $S$. Choose the orientation of $C$ that is compatible with the choice of side for $d\vec{A}$. If $\vec{F}$, $S$, and $C$ are "nice" (in a specific technical sense), then

$$\oiint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint_C \vec{F} \cdot d\vec{R}.$$

We give an outline of the proof in the following steps.

1. Use the definition of surface integral to write

   $$\oiint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \lim_{\Delta A \to 0} \sum_{j=1}^{n} \sum_{i=i}^{m} \vec{\nabla} \times \vec{F}(P_{ij}) \cdot \Delta\vec{A}_{ij}.$$

   Here, we think of breaking the surface $S$ into pieces $\Delta S_{ij}$ having area vectors $\Delta\vec{A}_{ij} = \Delta A_{ij} \hat{n}_{ij}$. (See Figure 5.) For each piece, we pick a point $P_{ij}$ at which to evaluate the vector field. Also, let $\Delta C_{ij}$ be the curve that forms the edge of the $ij$-th piece.

2. Substitute using Equation (6) to get

   $$\oiint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \lim_{\Delta A \to 0} \sum_{j=1}^{n} \sum_{i=i}^{m} \oint_{\Delta C_{ij}} \vec{F} \cdot d\vec{R}.$$

3. The sum of line integrals on the left is over curves $\Delta C_{ij}$ with lots of common edges. Using the result in the previous subsection, we see that all of the contributions from the interior edges cancel since each is traversed twice, once in each direction. The net result is from the contributions on the exterior edges which is equivalent to the line integral over the original curve $C$. That is,

   $$\sum_{j=1}^{n} \sum_{i=i}^{m} \oint_{\Delta C_{ij}} \vec{F} \cdot d\vec{R} = \oint_C \vec{F} \cdot d\vec{R}.$$

   Substituting this gives

   $$\oiint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint_C \vec{F} \cdot d\vec{R}.$$

   and we are done.

The technical details that are missing here justify the approximation we use in Step 2.

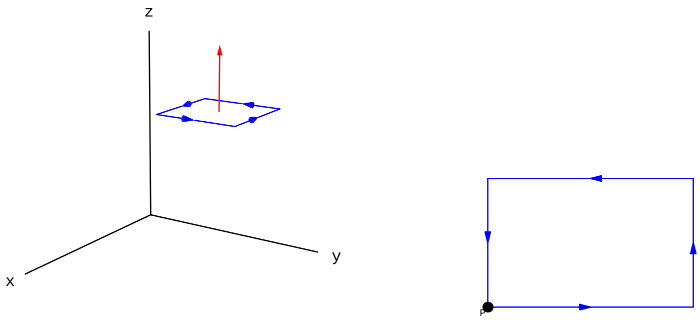


Figure 1. The small rectangle $\Delta C$ shown in perspective (left) and from above (right).

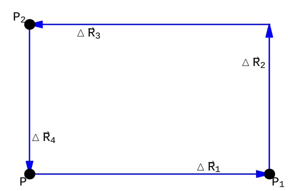


Figure 2. The points and edges of the small rectangle $\Delta C$.

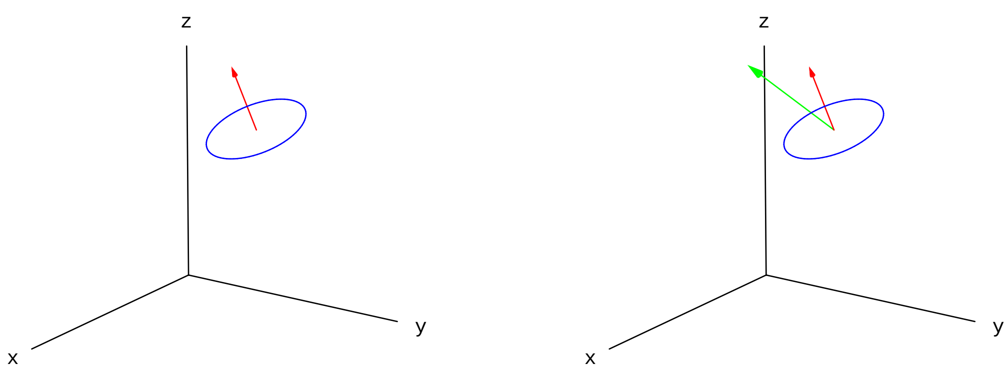


Figure 3. Left: A small flat loop $\Delta C$ with area vector $\Delta\vec{A} = \Delta A\,\hat{n}$ in red. Right: Same as left with the addition of the curl $curlF$ at $P$ (in green).

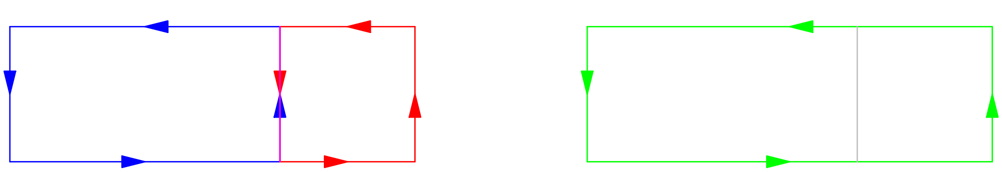

Figure 4. Two rectangular loops sharing a common edge.

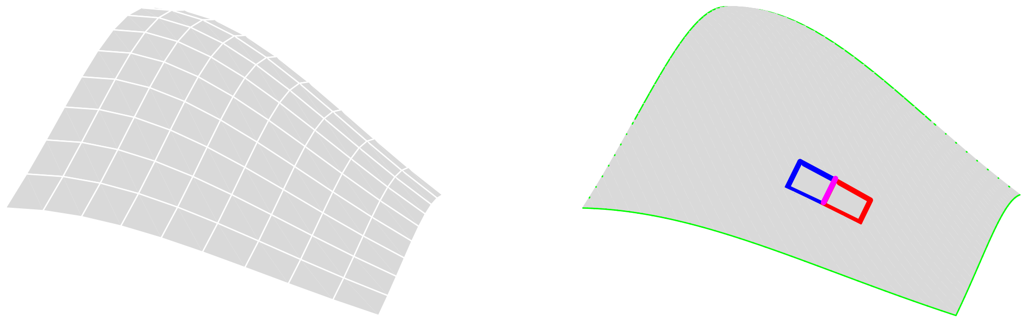

Figure 5. Left: The surface $S$ split into pieces $\Delta S_{ij}$. Right: Two adjacent pieces on the surface $S$.

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