A few problems on Stokes' Theorem

1. Let ΔC be a small, flat, closed loop in space centered at some specific point P. (You can think of a square or circle for simplicity). Let $\Delta\vec{A} = \Delta A\,\hat{n}$. In class, we argued that

$$\hat{n}\text{-component of } \vec{\nabla}\times\vec{F} \text{ at } P \;= (\vec{\nabla}\times\vec{F})\cdot\hat{n} \approx \frac{\oint_{\Delta C}\vec{F}\cdot d\vec{R}}{\Delta A}.$$

   Now think of a fluid flow interpretation of the vector field. That is, think of $\vec{F}$ as giving the fluid velocity at each point in space. Think of a paddlewheel or propellor inside the loop ΔC with the axis of the propellor aligned with $\hat{n}$. What does the $\hat{n}$-component of $\vec{\nabla}\times\vec{F}$ tell us about the rotation of this propellor? How do we orient the propellor (in relation to $\vec{\nabla}\times\vec{F}$) to get the fastest rotation rate? How do we orient the propellor to get no rotation? Out of this, make a statement about how we can interpret the *direction* of the curl vector at a specific point in space. Try to make an analogy here between the way we interpret gradient and this way of thinking about curl.

2. *(Note: The goal of this problem is to see how Green's Theorem is a special case of Stokes' Theorem.)* Let D be a bounded region in the xy-plane. Let C be the planar curve that forms the edge of D. Let $\vec{F}$ be a vector field of the form

$$\vec{F} = u(x,y)\,\hat{\imath} + v(x,y)\,\hat{\jmath} + 0\,\hat{k}.$$

   Now think of D as a surface in space (i.e., a flat surface that happens to sit in the xy-plane). Choose $d\vec{A}$ to point in the direction of the positive z-axis. Orient C in the counter-clockwise direction as seen from above the xy-plane. Now apply Stokes' Theorem to this surface and the vector field $\vec{F}$. Specifically, use what you know about $d\vec{A}$ and $\vec{\nabla}\times\vec{F}$ for this case in the surface integral that appears in Stokes' Theorem. Compare what you get with Green's Theorem as it is given on page 888 of the text.

3. Do Problems 1, 3, and 7 from Section 13.4 of the text. Each of these problems involves using Green's Theorem to trade a line integral in for a double integral. In general, it is easier to evaluate a double integral so this illustrates a practical use of Green's Theorem.