Solving the trigonometric equations in Problem 32 of Section 4.3

In Problem 32 of Section 4.3, you need to solve several trigonometric equations to get zeroes and critical points. The given function is $t(\theta) = \sin\theta - 2\cos\theta$ for $0 \le \theta \le 2\pi$. To get the zeroes of this function, you need to solve the equation

$$\sin\theta - 2\cos\theta = 0.$$

There are several ways you could get solutions, including using the **SOLVER** feature of your calculator. Here's an approach that is more direct.

First, do some algebra to rewrite the equation as

$$\frac{\sin\theta}{\cos\theta} = 2.$$

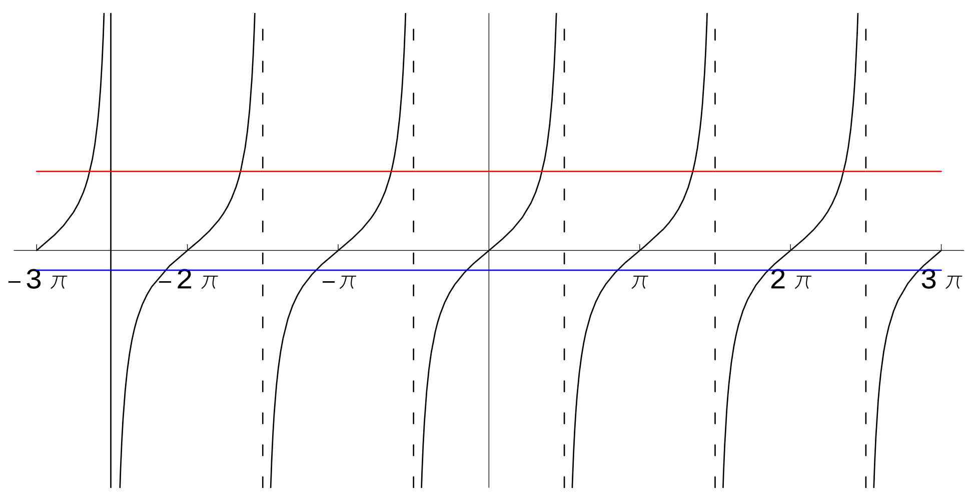
The expression on the left side is the definition of $\tan\theta$ so we can write the equation as

$$\tan\theta = 2.$$

Since 2 is not one of the values that show up on the table of inputs and outputs for the "nice" angles (0, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$), we're going to have be content with a decimal approximation. Using a calculator, we can get

$$\tan^{-1}(2) \approx 1.107.$$

However, this is not the only solution to the equation $\tan\theta = 2$. In fact, there are *infinitely many* solutions. To see this, look at the following plot.



This shows the graph of the tangent function (in black) along with the line $y = 2$ (in red). (Ignore the blue line for now.) The tangent function has vertical asymptotes at

$$\theta = \ldots, \frac{-5\pi}{2}, \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and is periodic with a period of π. The red line intersects the tangent graph infinitely many times (obviously not all shown in this plot). The inverse tangent function $\tan^{-1}$ gives outputs from the middle branch of the tangent so the outputs from your calculator will always be between $-\pi/2$ and $\pi/2$. The value 1.107 that we got above corresponds to the intersection of the red line with the tangent graph that is just to the left of the vertical asymptote at $\pi/2$. For our problem, we need to get all of the solutions of the equation $\tan\theta = 2$ that are in the interval $[0, 2\pi]$. From $\tan^{-1}(2) \approx 1.107$, we can get a second solution by adding π:

$$1.107 + \pi \approx 4.249.$$

We could continue to add or subtract π to get more solutions but only 1.107 and 4.249 fall into the interval $[0, 2\pi]$ given in this problem.

Later in this same problem, we need to solve the equation

$$\cos\theta + 2\sin\theta = 0.$$

We can follow the same strategy as before. First, rewrite the equation as

$$\tan\theta = -\frac{1}{2}.$$

Then use a calculator to get

$$\tan^{-1}\left(-\frac{1}{2}\right) \approx -0.464.$$

This corresponds to the intersection of the line $y = -1/2$ (the blue line in the plot) with the tangent graph just to the right of the vertical asymptote at $-\pi/2$. This value of -0.464 does not fall in the interval $[0, 2\pi]$. To get the solutions that do fall in this interval, we add π and then add π again. The two solutions for this problem are

$$-0.464 + \pi \approx 2.678 \qquad \text{and} \qquad -0.464 + 2\pi \approx 5.820.$$