Some vector problems

Physicists and chemists model a crystal as a three-dimensional lattice of points with a single atom centered at each point. To describe a lattice, pick a set of three vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ which have a right-hand orientation. The lattice can then be thought of as the set of all points with position vectors of the form $n_1\vec{v}_1 + n_2\vec{v}_2 + n_3\vec{v}_3$ where n_1, n_2, and n_3 are integers.

For a set of lattice vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, the *reciprocal lattice vectors* are defined as

$$\vec{k}_1 = \frac{\vec{v}_2 \times \vec{v}_3}{\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)}, \qquad \vec{k}_2 = \frac{\vec{v}_3 \times \vec{v}_1}{\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)}, \qquad \text{and} \qquad \vec{k}_3 = \frac{\vec{v}_1 \times \vec{v}_2}{\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)}.$$

These reciprocal lattice vectors are useful in analyzing the physical properties of crystals. Here we will look only at some of the mathematical results which follow from the definitions given above. The identities you will prove give some justification to the name "reciprocal" lattice vectors.

1. Give a geometric interpretation of each reciprocal lattice vector. That is, what geometric meaning can be assigned to the direction and magnitude of $\vec{k}_i$?

2. Show that $\vec{k}_1 \cdot \vec{v}_1 = 1$.

3. Show that $\vec{k}_1 \cdot (\vec{k}_2 \times \vec{k}_3) = \dfrac{1}{\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)}$.

   Hint: Use the "bac-cab" identity: $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$.

4. Give a geometric interpretation of the identity in 3.