Name ______________________________

MATH 302 Partial Differential Equations Fall 2011 Exam #6

Instructions: Do your own work. You may consult your class notes and the course text. Do not consult other sources. Do not discuss generalities or specifics of the exam with anyone except me.

Turn in a complete and concise write up of your work. Show enough detail so that a peer could follow your work (both computations and reasoning). All plots should be carefully drawn either by hand or printed from technology. If you want to include a visualization that cannot be printed (such as an animation), include it as an attachment in an email with "Math 302 Exam 6" as the subject line.

This exam has two problems with one worth 75 points and the other worth 25 points for a total of 100 points. You get to choose which of the problems is worth 75 points and which is worth 25 points. Make your choice clear in your submission.

The exam is due on Tuesday, November 22.

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1. The equation
$$-\left(p(x)y'\right)' + q(x)y = \lambda w(x)y$$
is a generalization of the Sturm-Liouville equation as given in Section 3.4. In addition to the assumptions about p and q for a regular SLP, assume that w is continuous and $w(x) > 0$ for x in $[a,b]$. The SL theorem holds for eigenvalues and eigenfunctions of this generalization with a small modification, namely that the relevant inner product is
$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx.$$
   (a) Prove that $\langle f, g\rangle$ as given above is, in fact, an inner product for $L^2[a,b]$.
   (b) Prove that if y_m and y_n are eigenfunctions corresponding to different eigenvalues λ_m and λ_n, then y_m and y_n are orthogonal with respect to this inner product.
   (c) Find the eigenvalues and eigenfunctions for the SL problem
$$\begin{aligned} (xy')' + \frac{\lambda}{x}y &= 0 \qquad \text{for } 1 < x < 2 \\ y(1) &= 0 \\ y(2) &= 0 \end{aligned}$$
   (d) Expand the function $f(x) = 1$ in terms of the eigenfunctions from Part 3.

   Hint: An *Euler equation* is a second-order ODE of the form
$$ax^2y'' + bxy' + cy = 0.$$
   Look for solutions in the form $y = x^r$ where r is a constant. If r turns out to have an imaginary part, you might want to use the fact that
$$x^{is} = e^{\ln(x^{is})} = e^{is\ln x} = \cos(s\ln x) + i\sin(s\ln x).$$

2. Consider a rod of length $l = 2$ meters made from material with diffusivity $k = 1.14 \times 10^{-4}$ m^2/s. The rod is thin and the lateral sides of the rod are insulated so heat can effectively only flow along one direction. One end of the rod is insulated and the other end is allowed to convect heat into the surrounding environment, which is at a constant temperature of 0°C. The rod is intially at a uniform temperature of $T_0 = 50$°C. A reasonable model for this situation is given by

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\begin{aligned}
u_t &= k u_{xx} && \text{for } 0 < x < l, \quad t > 0 \\
u_x(0,t) &= 0 && \text{for } t > 0 \\
u_x(l,t) &= -\alpha u(l,t) && \text{for } t > 0 \\
u(x,0) &= T_0 && \text{for } 0 < x < l
\end{aligned}
$$

with $\alpha = 1.0$ m^{-1}. Determine (at least approximately) how long it takes for the temperature at $x = 0$ to reach 10°C.