Regular Sturm-Liouville problems

A *Sturm-Liouville problem* consists of

$$\begin{aligned}
&\text{ODE:} && -\Big(p(x)y'(x)\Big)' + q(x)y(x) = \lambda y(x) && \text{for } a < x < b \\
&\text{BC1:} && \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\
&\text{BC2:} && \beta_1 y(b) + \beta_2 y'(b) = 0
\end{aligned}$$

A Sturm-Liouville problem is *regular* if

- p p', and q are continuous for $a \leq x \leq b$
- $p(x) > 0$ for $a \leq x \leq b$
- $(\alpha_1, \alpha_2) \neq (0,0)$ and $(\beta_1, \beta_2) \neq (0,0)$

Regular Sturm-Liouville problems

For a regular Sturm-Liouville problem:

1. There are countably many eigenvalues λ_n, $n = 1, 2, 3, \ldots$
2. Each eigenvalue is real.
3. There is a smallest eigenvalue and no largest eigenvalue so the eigenvalues can be ordered $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$.
4. For each eigenvalue, there is a one-dimensional eigenspace that is spanned by a single eigenfunction y_n.
5. The eigenfunction y_n has $n-1$ zeros for $a < x < b$.
6. The set of eigenfunctions $\{y_n | n = 1, 2, 3, \ldots\}$ is a maximal orthogonal set in $L^2[a, b]$ with respect to the inner product $< f, g >= \int_a^b f(x)g(x)\, dx$ and hence a basis.