ODE review problems

In the following problems, the independent variable is x and the dependent variable is y.

1. Find the general solution of the differential equation $y' = xy^2 + x$.

2. Consider the differential equation $y' = 3y + e^x$.

   (a) Find the general solution using the fact that the general solution to a first-order linear differential equation has the form $y = cy_h + y_p$ where c is a constant, y_h is a solution to the related homogeneous equation, and y_p is a solution to the nonhomogeneous equation.

   (b) Find the general solution using an integrating factor.

3. For each of the following, find the general solution for the given differential equation.

   (a) $y'' + 2y' - 15y = 0$

   (b) $y'' + 6y' + 13y = 0$

   (c) $x^2y'' - 3xy' + 3y = 0$
   Hint: Look for solutions in the form $y = x^m$ where m is constant.

4. The basic *hyperbolic functions* are defined as

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

   (a) Plot $\frac{1}{2}e^x$ and $\frac{1}{2}e^{-x}$ on the same plot. Use these to sketch $\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ on that plot. Finally, sketch $\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$ on the same plot.

   (b) Show that $\cosh^2 x - \sinh^2 x = 1$.

   (c) Show that $\text{Span}(e^x, e^{-x}) = \text{Span}(\cosh x, \sinh x)$.

   (d) Express the general solution of $y'' = 9y$ in terms of exponential functions. Find the specific solution that satisfies the initial conditions $y(0) = 5$ and $y'(0) = 0$.

   (e) Express the general solution of $y'' = 9y$ in terms of hyperbolic functions. Find the specific solution that satisfies the initial conditions $y(0) = 5$ and $y'(0) = 0$.