Gaussian functions and the error function

1. For this problem, do not use calculator or computing technology.
   (a) Plot several cycles of the function $f(x) = \sin(x)$ using appropriate multiples of π for the scale on the x-axis.
   (b) Use scaling and shifting to plot each of following: $\frac{1}{2}f(x)$, $2f(x)$, $f(\frac{1}{2}x)$, $f(2x)$, $f(x+\pi)$, and $f(x-\pi)$ for $f(x) = \sin x$.
   (c) Plot the function $f(x) = e^{-x^2}$ using calculus techniques to capture all of the essential features (such as local extremes, inflection points, and asymptotes).
   (d) Use scaling and shifting to plot each of following: $\frac{1}{2}f(x)$, $2f(x)$, $f(\frac{1}{2}x)$, $f(2x)$, $f(x+1)$, and $f(x-1)$ for $f(x) = e^{-x^2}$.

2. A *Gaussian function* has the form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

   with σ a positive constant. In probability, this is also referred to as a *Gaussian distribution* or a *normal distribution*. A graph of a Gaussian function is a *bell curve*.

   (a) Prove that every Gaussian function f is increasing for $x < 0$ and decreasing for $x > 0$.
   (b) Prove that every Gaussian function f is concave down for $|x| < \sigma$ and concave up for $|x| > \sigma$ (so f has inflection points at $x = \pm\sigma$).
   (c) Use the properties in (a) and (b) to sketch a plot of a generic Gaussian function. As part of this, choose useful scales in terms of σ for the axes.

3. For Gaussians functions as defined in Problem 2, prove that $\int_{-\infty}^{\infty} f(x)\,dx = 1$ for all $\sigma > 0$ by verifying or filling in the details of the following steps. That is, you should verify the reasoning and calculations in each step (including justifying each equality in strings of equalities) and you should fill in any omitted computational details.

   (a) For convenience, let $I = \int_{-\infty}^{\infty} f(x)\,dx$. Note that the value of I does not care about the name of the integration variable so we can also write $I = \int_{-\infty}^{\infty} f(y)\,dy$.
   (b) By symmetry, $\frac{1}{2}I = \int_0^{\infty} f(x)\,dx = \int_0^{\infty} f(y)\,dy$.
   (c) So, $\frac{1}{4}I^2 = (\frac{1}{2}I)(\frac{1}{2}I) = \int_0^{\infty} f(x)\,dx \cdot \int_0^{\infty} f(y)\,dy = \int_0^{\infty}\int_0^{\infty} f(x)f(y)\,dxdy$.
   (d) Substituting specific expressions for $f(x)$ and $f(y)$ and doing some rearranging, we have

$$\frac{1}{4}I^2 = \frac{1}{2\pi\sigma^2}\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)/2\sigma^2}\,dxdy$$

(e) The iterated integral in (d) corresponds to a double integral over the first quadrant of the xy-plane. Transforming to polar coordinates, we can express this as

$$\tfrac{1}{4}I^2 = \frac{1}{2\pi\sigma^2}\int_0^{\pi/2}\int_0^{\infty} e^{-r^2/2\sigma^2}\, r dr d\theta = \frac{1}{2\pi\sigma^2}\int_0^{\pi/2} d\theta \cdot \int_0^{\infty} e^{-r^2/2\sigma^2}\, r dr$$

(f) Evaluating the θ- and r-integrals (using a substitution for the r integral), we get

$$\tfrac{1}{4}I^2 = \frac{1}{2\pi\sigma^2}\cdot\frac{\pi}{2}\cdot\sigma^2 = \frac{1}{4}$$

(g) Thus, $I = \displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$.

4. The First Fundamental Theorem of Calculus tells us: *If f is continuous on $[a, b]$, then F defined as $F(x) = \displaystyle\int_a^x f(z)\, dz$ is an antiderivative for f on $[a, b]$. In other words,*

$$F'(x) = \frac{d}{dx}\int_a^x f(z)\, dz = f(x).$$

Note that we can think of $F(x)$ as the (signed) area under the graph of f from a to x. In many cases, this presents a convenient way to represent an antiderivative since there are many functions for which antiderivatives cannot be explicitly expressed in terms of elementary functions. One such example is the basic Gaussian function $f(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$. This function has antiderivatives but there is no formula to express these antiderivatives in terms of elementary functions. We can represent the antiderivatives as $\displaystyle\int_a^x \frac{1}{\sqrt{\pi}}e^{-z^2}\, dz$ for any constant a. Let's focus on the antiderivative corresponding to $a = 0$, so

$$F(x) = \frac{1}{\sqrt{\pi}}\int_0^x e^{-z^2}\, dz.$$

(a) Prove that $F(x) < 0$ for $x < 0$, $F(0) = 0$, and $F(x) > 0$ for $x > 0$.

(b) Prove that F is increasing for all x.

(c) Prove that F is concave up for $x < 0$ and concave down for $x > 0$.

(d) Prove that $\displaystyle\lim_{x\to-\infty} F(x) = -\frac{1}{2}$ and $\displaystyle\lim_{x\to\infty} F(x) = \frac{1}{2}$.

(e) Sketch a plot of $F(x)$ using the results you have proven and the idea that $F(x)$ is the (signed) area under the graph of $\dfrac{1}{\sqrt{\pi}}e^{-z^2}$ from 0 to x.

(f) The *error function* is defined as $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-z^2}\, dz$. Note that this is just twice $F(x)$. Use your results about F to sketch a plot of the error function.