Density

Most of us first learned about density as "mass divided by volume". This made sense in considering a specific object of uniform composition. We can separately measure the mass M and the volume V of the object and then compute density as M/V. We will use the symbol ρ (the lower case Greek letter "rho") to denote this density. A formula for density is thus $\rho = M/V$.

Exercise 1: *The density of aluminum is about* $2.7\,\mathrm{g/cm^3}$. *Determine the mass of an aluminum cube with sides of length* $2\,\mathrm{cm}$.

Some objects are more naturally measured in terms of length or area rather than volume. For something like a rope or rod, we can use *length density* defined as "mass per length". We will typically use the symbol λ (the lower case Greek letter "lambda") to denote length density.

Exercise 2: *A particular type of rope has a length density of* $\lambda = 0.15$ *kg per meter. What is the mass of a 3 meter piece of this rope?*

For something like a sheet of paper or a piece of plywood, we can use *area density* defined as "mass per area". To denote area density, we will generally use σ (the lower case Greek letter "sigma").

Exercise 3: *A standard type of newsprint has an area density of* $48.8\,\mathrm{g/m^2}$. *Determine the mass of a roll of this newsprint that is 2 meters wide and 100 meters long.*

You will encounter many references to density that do not specifically refer to length, area, or volume. You will need to use context and units to determine which type of density is in play.

Density need not refer to *mass*. Other quantities for which density is relevant include number, cost, charge, and probability. For example, the advertised cost of flooring is often given in dollars per square foot so a particular carpet might be listed as $4.25 per square foot. This is an area density for cost.

Exercise 4: *A circular lid of radius* 3 *inches is made from material that* \$0.04 *per square inch. Determine the total cost of the material for the lid.*

In each of the exercises above, the density is uniform throughout the relevant region si the total is easily computed as the density multiplied by a measure (length, area, or volume) of the region. An object with uniform composition throughout has the same mass density at each point. Likewise, if charge is spread uniformly throughout a region, the charge density is the same at each point. More generally, density varies from point to point throughout the relevant region. If mass or charge is not distributed uniformly, then the density varies from point to point and computing a total becomes more interesting.

Our first step in working with a nonuniform density is to find useful mathematical descriptions. We'll start here by examining nonuniform density on a line segment. (Later in the course, we'll generalize to nonuniform density on more interesting regions.) The following example illustrates several ways to describe a nonuniform density.

Example 1

Charge is distributed along a line segment of length L so that the length charge density is proportional to the distance from one end of the segment. Let λ_0 denote the maximum charge density. Use a picture, a graph, and a formula to describe the density function for this distribution.

We begin by constructing a picture using shading to illustrate varying density as shown in Figure 1. Here, we choose a coordinate axis so that one end of the segment is at the origin $x = 0$ and the other end is at $x = L$. In this case, the density values range from $\lambda = 0$ at $x = 0$ to the maximum value $\lambda = \lambda_0$ at $x = L$. In this case, all values of λ are positive, so we choose a grayscale shading in which white represents a density of 0 and black represents the maximum density λ_0.

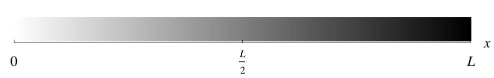


Figure 1: Schematic picture of the charge distribution for Example 1.

This type of *density plot* gives us one view of the charge distribution. We can convey the same information by plotting the density as a function of position (measured using the coordinate system described above) as shown in Figure 2.

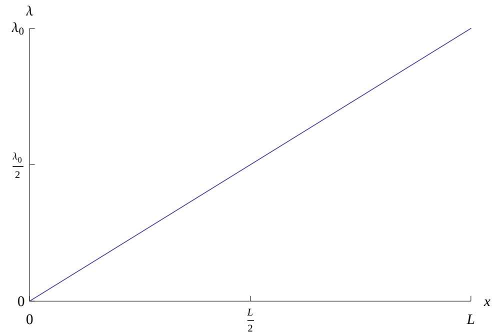


Figure 2: Graph of the charge density for Example 1.

To build a formula for density λ as a function of position x, we note that λ is proportional to the distance from $x = 0$ and x measures that distance so we can introduce a proportionality constant k and write

$$\lambda = kx.$$

To determine the value of k, we note that $\lambda = \lambda_0$ for $x = L$. Substituting these specific values in the proportionality relation gives us

$$\lambda_0 = kL. \tag{1}$$

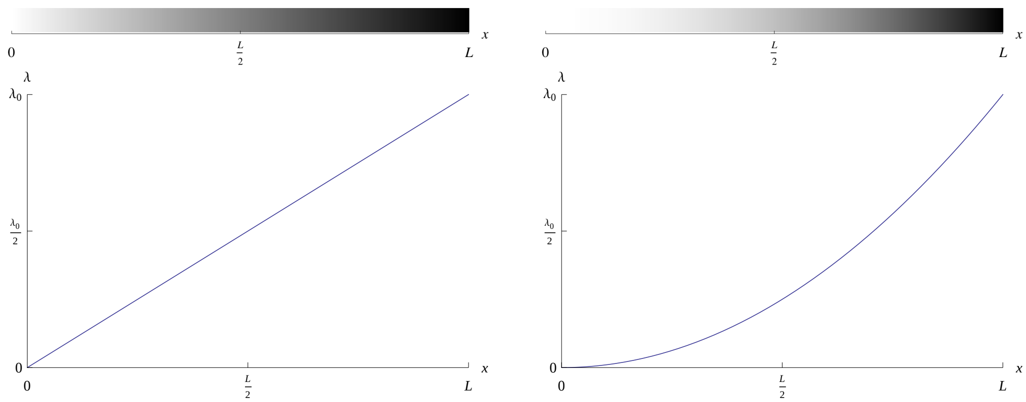
Solving, we find $k = \lambda_0/L$ so we can substitute this into (1) to get

$$\lambda = \frac{\lambda_0}{L}x \tag{2}$$

As a consistency check, note that our relation gives the correct values of $\lambda = 0$ for $x = 0$ and $\lambda = \lambda_0$ for $x = L$.

For another consistency check, we can look at units. In SI units, length has units of meters (m) and charge has units of Coulombs (C) so length charge density has units of C/m. So, the variable λ on the left side of (2) has units of C/m. On the right side of (2), the parameter λ_0 has units of C/m while the variable x and the parameter L have units of m. So, the units on the right side simplify to C/m in agreement with the units on the left side.

In the previous example, we gave two graphical ways of describing a specific nonuniform density distribution: a density plot and a graph of density as a function of position. A density plot provides useful qualitative information but is difficult to read quantitatively. As a second example, consider a situation in which charge is distributed on a line segment with the length charge density proportional to the *square* of the distance from one end. A density plot and graph of density versus position for this distribution is shown in Figure 3(b) with the previous density plot and graph repeated on the Figure 3(a) for comparison.



(a) Density proportional to distance from $x = 0$.

(b) Density proportional to square of distance from $x = 0$.

Figure 3: Comparing two charge distributions.

For a density with both positive and negative values, we could use a grayscale shading or we could use two colors, with one color representing positive values and the other representing negative values. We illustrate this in the next example.

Example 2

Charge is distributed along a line segment of length L so that the length charge density varies sinusoidally through one cycle starting with value $\lambda = 0$ at one end. Let λ_0 denote the maximum charge density. Use a picture, a graph, and a formula to describe the density function for this distribution.

In Figure 4, we show a density plot and a graph of density versus position. In the density plot, white presents $\lambda = 0$, the darkest blue represents $\lambda = \lambda_0$, and the darkest red represents $\lambda = -\lambda_0$.

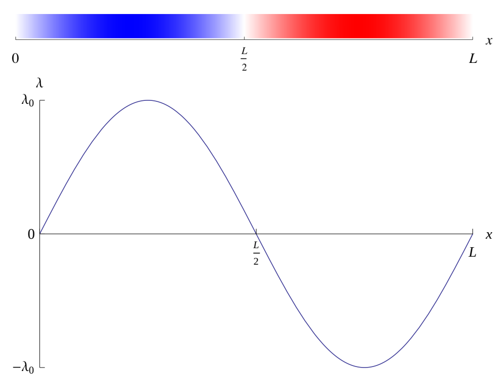


Figure 4: Charge distribution for Example 2.

To determine a formula, we can use a sine function with suitable choices of amplitude and period. In this case, the amplitude is λ_0 and the period is L so we have

$$\lambda = \lambda_0 \sin\left(\frac{2\pi}{L} x\right).$$

As a consistency check, note that the formula gives $\lambda = 0$ for $x = 0$ and $\lambda = 0$ for $x = L$.

Knowing a length charge density along a line segment, we can compute the total charge Q using integration. Suppose we have charge distributed on a line segment of length L with length charge density λ as shown in Figure 5. We let x measure the position along the segment starting with $x = 0$ at one end. For an infinitesimal piece of the segment having length dx, the corresponding infinitesimal charge is $\lambda\, dx$. (To be more precise, we could use $\lambda(x)$ to indicate a specific value of the charge density for the specific position x.)

Figure 5: An infinitesimal piece of a typical segment.

We get the total charge Q by summing up these infinitesimal contributions. We express this as the integral

$$Q = \int_0^L \lambda\, dx. \tag{3}$$

To evaluate the integral, we need a specific charge density function as illustrated in the next example.

Example 3

Compute the total charge for the situation of Example 1 in which charge is distributed along a line segment of length L so that the length charge density is proportional to the distance from one end of the segment with λ_0 denoting the maximum charge density.

In Example 1, we found a formula for the length charge density to be

$$\lambda = \frac{\lambda_0}{L} x.$$

Using this in (3), we have

$$Q = \int_0^L \frac{\lambda_0}{L} x\, dx.$$

Use the constant factor property of definite integrals and the Fundamental Theorem of Calculus, we get

$$Q = \frac{\lambda_0}{L} \int_0^L x\, dx = \frac{\lambda_0}{L} \frac{1}{2} x^2 \bigg|_0^L = \frac{1}{2} \frac{\lambda_0}{L} L^2 = \frac{1}{2} \lambda_0 L.$$

The final expression has the correct units since λ_0 has units of C/m while L has units of m so the product has units of C. The expression also makes intuitive sense in the following way: Imagine redistributing the charge by moving charge from the right half of the segment to the left half. Since the charge increases linearly with distance from the left end, we can redistribute the charge to get a uniform distribution with length charge density equal to $\frac{1}{2}\lambda_0$. The total charge for this configuration would be $\frac{1}{2}\lambda_0 L$. This matches our result above.

If the region in question is two-dimensional, then computing a total from a density involves constructing and evaluating a double integral (or, more generally, a surface integral). If σ represents an area charge density for a surface S, then the total charge Q is given by

$$Q = \iint_S \sigma \, dA.$$

In similar fashion, computing a total from a volume density ρ generally involves constructing and evaluating a triple integral over the relevant three-dimensional region D:

$$Q = \iiint_D \rho \, dV$$

Example 4

Charge is distributed on a square of side length L so that the area charge density is proportional to the square of the distance from one corner, reaching a maximum value σ_0 at the opposite corner. Use a picture, a graph, and a formula to describe the density function for this distribution. Then, compute the total charge.

To start, we set up a cartesian coordinate system with origin on the corner of the square at which the density is zero with axes running parallel to the sides of the square as shown on the left in Figure 6. At a generic point (x, y), the distance to the origin is given by $x^2 + y^2$. Since the density is proportional to this distance squared, we can introduce a proportionality constant k and write

$$\sigma = k(x^2 + y^2).$$

The corner farthest from the origin has coordinates (L, L) so we know that

$$\sigma_0 = k(L^2 + L^2) = 2kL^2.$$

Solving, we find $k = \sigma_0/2L^2$. Substituting this into the expression above gives

$$\sigma = \frac{\sigma_0}{2L^2}(x^2 + y^2).$$

A plot of σ as a function of x and y is shown on the right in FIgure 6.

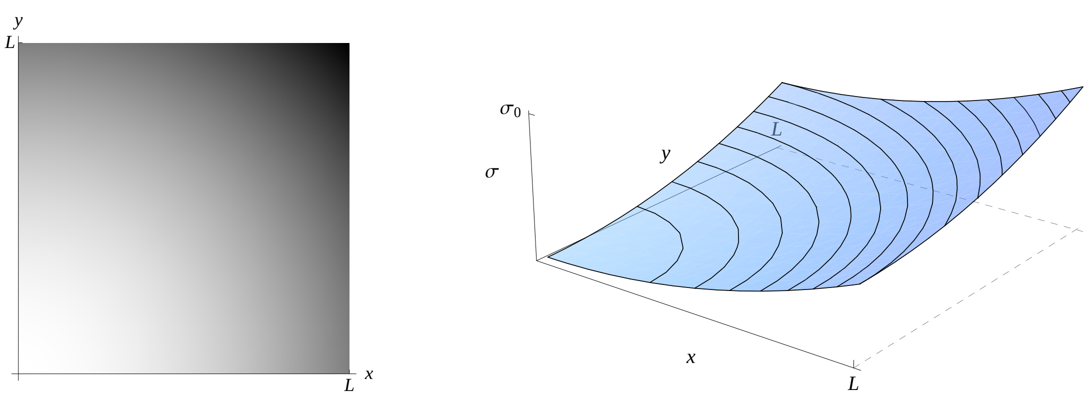


Figure 6: Charge distribution for Example .

To compute the total charge, we start with the double integral

$$Q = \iint\limits_{\text{square}} \sigma \, dA.$$

The equivalent iterated integral is

$$Q = \int_0^L \int_0^{2L^2} \frac{\sigma_0}{2L}(x^2 + y^2)\, dxdy.$$

Evaluating this integral in steps we have

$$\begin{aligned}
Q &= \frac{\sigma_0}{2L^2}\int_0^L \int_0^L (x^2+y^2)\, dxdy = \frac{\sigma_0}{2L^2}\int_0^L (x^2 y + \frac{1}{3}y^3\Big|_0^L \, dx \\
&= \frac{\sigma_0}{2L^2}\int_0^L (Lx^2 + \frac{1}{3}L^3)\, dx = \frac{\sigma_0}{2L^2}(\frac{1}{3}Lx^3 + \frac{1}{3}L^3 x\Big|_0^L \\
&= \frac{\sigma_0}{2L^2}(\frac{1}{3}L^4 + \frac{1}{3}L^4) = \frac{\sigma_0}{2L^2}\frac{2}{3}L^4 = \frac{1}{3}\sigma_0 L^2.
\end{aligned}$$

The final expression has the correct units since σ_0 has units of $\mathrm{C/m^2}$ while L^2 has units of m^2 so the product has units of C. The expression is also consistent with the upper bound of $\sigma_0 L^2$ that we get by thinking about a uniform distribution with the maximum charge density σ_0 throughout the square.

Problems

1. Consider a situation in which charge is distributed on a line segment of length L with length charge density proportional to the square of the distance from one end. Let λ_0 denote the maximum charge density at the other end. A schematic picture and a graph of this charge density is shown in Figure 3(b).

   (a) Construct a formula for the length charge density.

   (b) Compute the total charge on the segment.

2. Charge is distributed on a line segment of length L with length charge density proportional to the the distance from the center of the segment. Let λ_0 denote the maximum charge density.

   (a) Sketch a schematic picture using shading to represent density.

   (b) Set up a coordinate axis along the line segment with the origin at one end of the segment. Sketch a plot of density as a function of position as measured on this coordinate axis.

   (c) Set up a coordinate axis along the line segment with the origin at the center of the segment. Sketch a plot of density as a function of position as measured on this coordinate axis.

   (d) Construct a formula for the density as a function of position. For this, you can use either of the coordinate systems from (b) and (c).

   (e) Compute the total charge on the segment.

3. Repeat the steps of the previous problem for the situation in which the length charge density is proportional to the square of the distance from the center of the segment.

4. Consider a thin rectangular plate of dimensions L by W with uniform thickness. The materials composing the plate vary from point to point so that the area mass density is proportional to the square of the distance from the center of the plate, reaching a maximum of σ_0 at each of the four corners. Compute the total mass M of the plate.

5. Charge is distributed on an isosceles triangle of height H and base length B so that the area charge density is proportional to the distance from the base, reaching a maximum of σ_0 at the vertex opposite the base. Compute the total charge Q.

6. Charge is distributed throughout a rectangular region of space having dimension L by W by H so that the volume charge density is proportional to the square of the distance from one corner, reaching a maximum of δ_0 at the far corner. Compute the total charge Q.