The delta function

In exploring the limit as $t \to 0^+$ of the fundamental solution to the heat equation, we encountered a new "function". Recall that the fundamental solution is

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}.$$

In analyzing the limit as $t \to 0^+$, we found

$$\lim_{t\to 0^+} G(x,t) = 0 \quad \text{for } x \neq 0 \qquad \text{and} \qquad \lim_{t\to 0^+} G(0,t) = \infty.$$

We also showed that

$$\int_{-\infty}^{\infty} G(x,t)\, dt = 1 \qquad \text{for } t > 0.$$

These results motivated us to define the *delta function* δ as having the properties

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

We can think of $\delta(x)$ as a "function" with value 0 at all but one point and an area of 1.

In terms of the delta function, we can express the limit of the fundamental solution as

$$\lim_{t\to 0^+} G(x,t) = \delta(x).$$

We could also write this more compactly as

$$G(x,0^+) = \delta(x).$$

In other words, the delta function is the initial condition from which the fundamental solution evolves for $t > 0$.

In physical terms, we can think of $\delta(x)$ as representing 1 unit of heat energy all concentrated at $x = 0$. For this configuration, the heat energy density u is zero for $x \neq 0$ and infinite for $x = 0$. (To be more precise, the heat energy density would be undefined for $x = 0$. We'll think of the "value" as ∞ for convenience.) We represent this graphically with an arrow at $x = 0$ as shown in Figure 1. The number at the top of the arrow gives the total heat energy associated with the delta function. Note that this value does not belong on the u axis since it is a total energy rather than an energy *density*.

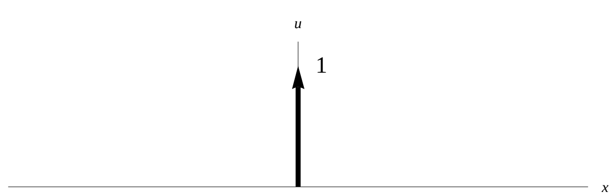

Figure 1: A graphical representation of $\delta(x)$.

We can scale and shift the delta function to represent any amount of heat energy concentrated at any point. For example, 3 units of heat energy at $x = -2$ and 5 unit of heat energy at $x = 1$ is represented symbolically by $3\delta(x+2) + 5\delta(x-1)$ and graphically as shown in Figure 2. Note that each arrow height is proportional to the total energy at that point. In terms of area, we can think of the arrow at $x = -2$ as having area 3 and the arrow at $x = 1$ as having area 5.

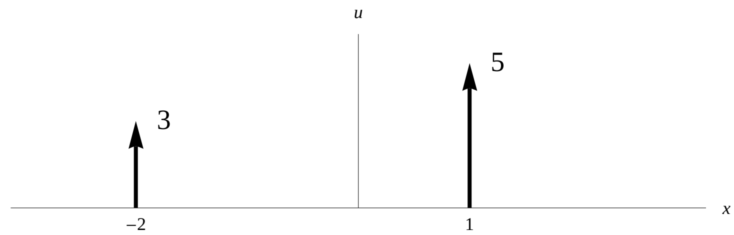

Figure 2: A graphical representation of $3\delta(x+2) + 5\delta(x-1)$.

Integrals involving delta functions are easy to evaluate. Let f be any function and consider the integral

$$\int_{-\infty}^{\infty} f(z)\delta(z-a)\,dz$$

where a is a constant. The integrand here is the product of $f(z)$ and $\delta(z-a)$. For $z \neq a$, we have $\delta(z-a) = 0$ so $f(z)\delta(z-a) = 0$. For $z = a$, the delta function is multiplied by $f(a)$. So, the product is a delta function of area $f(a)$. This is represented graphically in Figure 3.

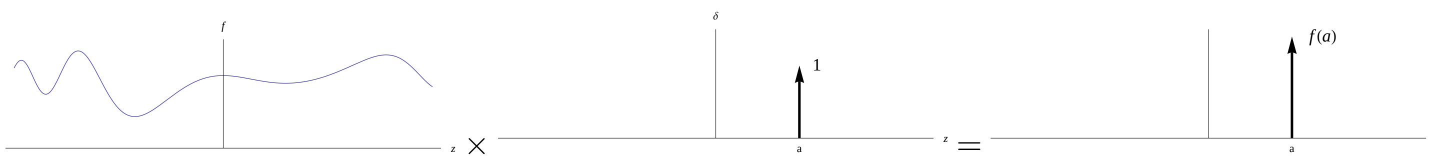

Figure 3: A graphical representation of $f(z)\delta(z-a)$.

Since the integral gives us the total area of the scaled delta function, we have the result

$$\int_{-\infty}^{\infty} f(z)\delta(z-a)\,dz = f(a)$$

For example, we can use this result to compute

$$\int_{-\infty}^{\infty} \sin(z)\delta(z-\tfrac{\pi}{4})\,dz = \sin(\tfrac{\pi}{4}) = \tfrac{\sqrt{2}}{2}.$$

Since all of the area for a delta function is at one point, we can generalize this integral result. If a is in the interval $[\alpha, \beta]$, then

$$\int_{\alpha}^{\beta} f(z)\delta(z-a)\,dz = f(a).$$

If a is *not* in the interval $[\alpha, \beta]$, then

$$\int_{\alpha}^{\beta} f(z)\delta(z-a)\,dz = 0.$$