

### Describing and integrating nonuniform density on a line segment

An object with uniform composition throughout has the same mass density at each point. Likewise, if charge is spread uniformly throughout a region, the charge density is the same at each point. Computing a total from a uniform density requires only that we multiply the uniform density value by the relevant measure of the region (length, area, or volume). If mass or charge is not distributed uniformly, then the density varies from point to point and computing a total becomes more interesting.

Our first step in working with a nonuniform density is to find useful mathematical descriptions. We'll start here by examining nonuniform density on a line segment. (Later in the course, we'll generalize to nonuniform density on more interesting regions.) The following example illustrates several ways to describe a nonuniform density.

#### Example 1

*Charge is distributed along a line segment of length  $L$  so that the length charge density is proportional to the distance from one end of the segment. Let  $\lambda_0$  denote the maximum charge density. Use a picture, a graph, and a formula to describe the density function for this distribution.*

We begin by constructing a picture using shading to illustrate varying density as shown in Figure 1. Here, we choose a coordinate axis so that one end of the segment is at the origin  $x = 0$  and the other end is at  $x = L$ . In this case, the density values range from  $\lambda = 0$  at  $x = 0$  to the maximum value  $\lambda = \lambda_0$  at  $x = L$ . In this case, all values of  $\lambda$  are positive, so we choose a grayscale shading in which white represents a density of 0 and black represents the maximum density  $\lambda_0$ .

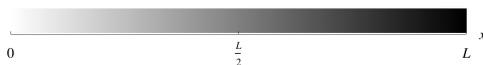


Figure 1: Schematic picture of the charge distribution for Example 1.

This type of *density plot* gives us one view of the charge distribution. We can convey the same information by plotting the density as a function of position (measured using the coordinate system described above) as shown in Figure 2.

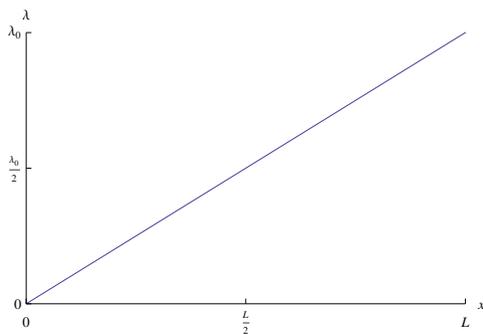


Figure 2: Graph of the charge density for Example 1.

To build a formula for density  $\lambda$  as a function of position  $x$ , we note that  $\lambda$  is proportional to the distance from  $x = 0$  and  $x$  measures that distance so we can introduce a proportionality constant  $k$  and write

$$\lambda = kx.$$

To determine the value of  $k$ , we note that  $\lambda = \lambda_0$  for  $x = L$ . Substituting these specific values in the proportionality relation gives us

$$\lambda_0 = kL. \tag{1}$$

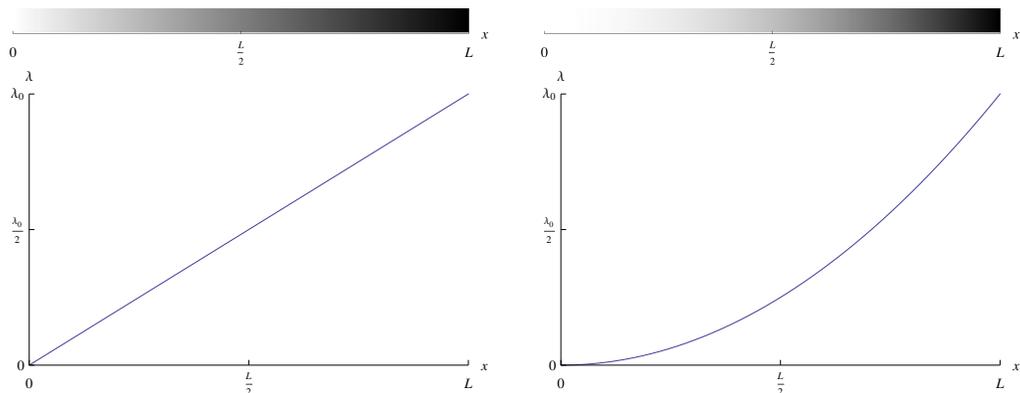
Solving, we find  $k = \lambda_0/L$  so we can substitute this into (1) to get

$$\lambda = \frac{\lambda_0}{L}x \tag{2}$$

As a consistency check, note that our relation gives the correct values of  $\lambda = 0$  for  $x = 0$  and  $\lambda = \lambda_0$  for  $x = L$ .

For another consistency check, we can look at units. In SI units, length has units of meters (m) and charge has units of Coulombs (C) so length charge density has units of C/m. So, the variable  $\lambda$  on the left side of (2) has units of C/m. On the right side of (2), the parameter  $\lambda_0$  has units of C/m while the variable  $x$  and the parameter  $L$  have units of m. So, the units on the right side simplify to C/m in agreement with the units on the left side.

In the previous example, we gave two graphical ways of describing a specific nonuniform density distribution: a density plot and a graph of density as a function of position. A density plot provides useful qualitative information but is difficult to read quantitatively. As a second example, consider a situation in which charge is distributed on a line segment with the length charge density proportional to the *square* of the distance from one end. A density plot and graph of density versus position for this distribution is shown in Figure 3(b) with the previous density plot and graph repeated on the Figure 3(a) for comparison.



(a) Density proportional to distance from  $x = 0$ . (b) Density proportional to square of distance from  $x = 0$ .

Figure 3: Comparing two charge distributions.

For a density with both positive and negative values, we could use a grayscale shading or we could use two colors, with one color representing positive values and the other representing negative values. We illustrate this in the next example.

**Example 2**

Charge is distributed along a line segment of length  $L$  so that the length charge density varies sinusoidally through one cycle starting with value  $\lambda = 0$  at one end. Let  $\lambda_0$  denote the maximum charge density. Use a picture, a graph, and a formula to describe the density function for this distribution.

In Figure 4, we show a density plot and a graph of density versus position. In the density plot, white presents  $\lambda = 0$ , the darkest blue represents  $\lambda = \lambda_0$ , and the darkest red represents  $\lambda = -\lambda_0$ .

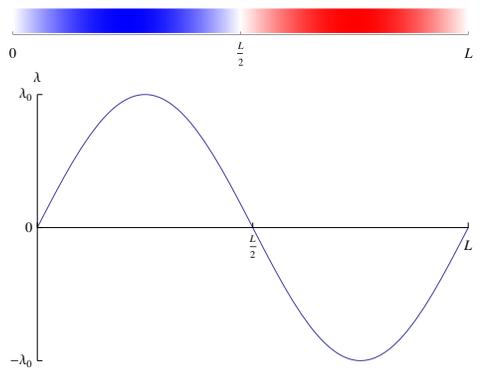


Figure 4: Charge distribution for Example 2.

To determine a formula, we can use a sine function with suitable choices of amplitude and period. In this case, the amplitude is  $\lambda_0$  and the period is  $L$  so we have

$$\lambda = \lambda_0 \sin\left(\frac{2\pi}{L}x\right).$$

As a consistency check, note that the formula gives  $\lambda = 0$  for  $x = 0$  and  $\lambda = 0$  for  $x = L$ .

Knowing a length charge density along a line segment, we can compute the total charge  $Q$  using integration. Suppose we have charge distributed on a line segment of length  $L$  with length charge density  $\lambda$  as shown in Figure 5. We let  $x$  measure the position along the segment starting with  $x = 0$  at one end. For an infinitesimal piece of the segment having length  $dx$ , the corresponding infinitesimal charge is  $\lambda dx$ . (To be more precise, we could use  $\lambda(x)$  to indicate a specific value of the charge density for the specific position  $x$ .)



Figure 5: An infinitesimal piece of a typical segment.

We get the total charge  $Q$  by summing up these infinitesimal contributions. We express this as the integral

$$Q = \int_0^L \lambda dx. \quad (3)$$

To evaluate the integral, we need a specific charge density function as illustrated in the next example.

**Example 3**

Compute the total charge for the situation of Example 1 in which charge is distributed along a line segment of length  $L$  so that the length charge density is proportional to the distance from one end of the segment with  $\lambda_0$  denoting the maximum charge density.

In Example 1, we found a formula for the length charge density to be

$$\lambda = \frac{\lambda_0}{L}x.$$

Using this in (3), we have

$$Q = \int_0^L \frac{\lambda_0}{L}x dx.$$

Use the constant factor property of definite integrals and the Fundamental Theorem of Calculus, we get

$$Q = \frac{\lambda_0}{L} \int_0^L x dx = \frac{\lambda_0}{L} \frac{1}{2}x^2 \Big|_0^L = \frac{1}{2} \frac{\lambda_0}{L} L^2 = \frac{1}{2} \lambda_0 L.$$

The final expression has the correct units since  $\lambda_0$  has units of C/m while  $L$  has units of m so the product has units of C. The expression also makes intuitive sense in the following way: Imagine redistributing the charge by moving charge from the right half of the segment to the left half. Since the charge increases linearly with distance from the left end, we can redistribute the charge to get a uniform distribution with length charge density equal to  $\frac{1}{2}\lambda_0$ . The total charge for this configuration would be  $\frac{1}{2}\lambda_0 L$ . This matches our result above.

### Problems: Describing and integrating nonuniform density on a line segment

1. Consider a situation in which charge is distributed on a line segment of length  $L$  with length charge density proportional to the square of the distance from one end. Let  $\lambda_0$  denote the maximum charge density at the other end. A schematic picture and a graph of this charge density is shown in Figure 3(b).
  - (a) Construct a formula for the length charge density.
  - (b) Compute the total charge on the segment.
2. Charge is distributed on a line segment of length  $L$  with length charge density proportional to the the distance from the center of the segment. Let  $\lambda_0$  denote the maximum charge density.
  - (a) Sketch a schematic picture using shading to represent density.
  - (b) Set up a coordinate axis along the line segment with the origin at one end of the segment. Sketch a plot of density as a function of position as measured on this coordinate axis.
  - (c) Set up a coordinate axis along the line segment with the origin at the center of the segment. Sketch a plot of density as a function of position as measured on this coordinate axis.
  - (d) Construct a formula for the density as a function of position. For this, you can use either of the coordinate systems from (b) and (c).
  - (e) Compute the total charge on the segment.
3. Repeat the steps of the previous problem for the situation in which the length charge density is proportional to the square of the distance from the center of the segment.
4. Charge is distributed on a line segment so that the length charge density is zero at the center, negative on one half of the segment, and positive on the other half. On each half, the magnitude of the charge density increases with distance from the center.
  - (a) For this situation, sketch a schematic picture, make a graph, and find a formula for the density as a function of position.
  - (b) Without doing a calculation, deduce the total charge on the segment.
5. Charge is distributed on a line segment of length  $L$  with length charge density increasing exponentially from a value of  $\lambda_0$  at one end to a value of  $2\lambda_0$  at the other end.
  - (a) For this situation, sketch a schematic picture, make a graph, and find a formula for the density as a function of position.
  - (b) Compute the total charge on the segment.