A linearization example

Problem: Consider the function $f(x,y) = x^2y^3$ for the input $(3,1)$.

(a) Compute the linearization of $f(x,y)$ based at $(3,1)$.

(b) Find an upper bound on the error $E(x,y) = f(x,y) - L(x,y)$ that is valid for the rectangle with $2 < x < 4$ and $0 < y < 2$ centered at $(3,1)$.

(c) Find an upper bound on the error $E(x,y) = f(x,y) - L(x,y)$ that is valid for the rectangle with $2.9 < x < 3.1$ and $0.9 < y < 1.1$ centered at $(3,1)$.

Solution for (a)

- evaluate f at the given input: $f(3,1) = (3)^2(1)^3 = 9$
- compute the partial derivatives and evaluate each at the given input

$$f_x(x,y) = 2xy^3 \qquad \text{so} \qquad f_x(3,1) = (2)(3)(1)^3 = 6$$

$$f_y(x,y) = 3x^2y^2 \qquad \text{so} \qquad f_y(3,1) = (3)(3)^2(1)^2 = 27$$

- substitute these values into the definition of the linearization function

$$\begin{aligned} L(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0)\,(y-y_0) \\ &= f(3,1) + f_x(3,1)\,(x-3) + f_y(3,1)\,(y-1) \\ &= 9 + 6(x-3) + 27(y-1) \end{aligned}$$

$$\boxed{L(x,y) = 9 + 6(x-3) + 27(y-1)}$$

Solution for (b)

- will use the following result

If M is an upper bound on $|f_{xx}|$, $|f_{yy}|$ and $|f_{xy}|$ for all (x,y) in a rectangle with $x_0 - a < x < x_0 + a$ and $y_0 - b < y < y_0 + b$, then

$$|E(x,y)| \le \frac{1}{2}M\big(|x-x_0| + |y-y_0|\big)^2$$

for all (x,y) in that rectangle

- compute the second partial derivatives

$$f_{xx}(x,y) = 2y^3 \qquad f_{yy}(x,y) = 6x^2y \qquad f_{xy}(x,y) = 6xy^2$$

- all three of these increase with both x and y so maximum values in the rectangle with $2 < x < 4$ and $0 < y < 2$ are at $(x,y) = (4,2)$

$$f_{xx}(4,2) = 2(2)^3 = 16 \qquad f_{yy}(4,2) = 6(4)^2(2) = 192 \qquad f_{xy}(4,2) = 6(4)(2)^2 = 96$$

- from these, see that $M = 192$ is an appropriate choice so have

$$|E(x,y)| \le \frac{1}{2}(192)\big(|x-3| + |y-1|\big)^2 = 96\big(|x-3| + |y-1|\big)^2$$

- to get a constant upper bound, go to one corner of the rectangle such as $(x,y) = (4,2)$ to get

$$|E(x,y)| \le 96\big(|4-3| + |2-1|\big)^2 = 96(1+1)^2 = 384$$

- so, for all (x,y) in the rectangle with $2 < x < 4$ and $0 < y < 2$, the maximum error in using $L(x,y) = 9 + 6(x-3) + 27(y-1)$ as an approximation for $f(x,y) = x^2y^3$ is 384
- on this rectangle, can be a lot of error in using L as an approximation for f

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Solution for (c)

- can use much of the work from (b)
- have computed the second partial derivatives

$$f_{xx}(x,y) = 2y^3 \qquad f_{yy}(x,y) = 6x^2y \qquad f_{xy}(x,y) = 6xy^2$$

- all three of these increase with both x and y so maximum values in the rectangle with $2.9 < x < 3.1$ and $0.9 < y < 1.1$ are at $(x,y) = (3.1, 1.1)$

$$f_{xx}(3.1,1.1) = 2(1.1)^3 = 2.662 \qquad f_{yy}(3.1,1.1) = 6(3.1)^2(1.1) = 63.426$$

$$f_{xy}(3.1,1.1) = 6(3.1)(1.1)^2 = 22.506$$

- from these, see that $M = 63.426$ is an appropriate choice so have

$$|E(x,y)| \le \frac{1}{2}(63.426)\big(|x-3| + |y-1|\big)^2 = 31.713\big(|x-3| + |y-1|\big)^2$$

- to get a constant upper bound, go to one corner of the rectangle such as $(x,y) = (3.1, 1.1)$ to get

$$|E(x,y)| \le 31.713\big(|3.1-3| + |1.1-1|\big)^2 = 31.713(0.1+0.1)^2 = 1.269$$

- so, for all (x,y) in the rectangle with $2.9 < x < 3.1$ and $0.9 < y < 1.1$, the maximum error in using $L(x,y) = 9 + 6(x-3) + 27(y-1)$ as an approximation for $f(x,y) = x^2y^3$ is 1.269
- the fact that this constant upper bound on the error is much smaller than in (b) is consistent with the fact that the error generally increases with distance away from the base point (x_0, y_0)