Fundamental Theorem for Definite Integrals

If $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

By substituting, we can also write the conclusion as

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Note: In the above and following theorems, a hypothesis on continuity of the integrand is omitted in order to focus on other details. In the following, a hypothesis on "niceness" of the relevant region is also omitted for the same reason.

Fundamental Theorem for Line Integrals

Let C be a curve that starts at A and ends at B. If $\vec{\nabla} V = \vec{F}$, then

$$\int_C \vec{F} \cdot d\vec{r} = V(B) - V(A).$$

By substituting, we can also write the conclusion as

$$\int_C \vec{\nabla} V \cdot d\vec{r} = V(B) - V(A).$$

The Divergence Theorem

Let D be a solid region with the closed surface S as the edge of D and area element vectors $d\vec{A}$ for S oriented outward. If $\vec{\nabla} \cdot \vec{F} = f$, then

$$\iiint_D f \, dV = \oiint_S \vec{F} \cdot d\vec{A}.$$

By substituting, we can also write the conclusion as

$$\iiint_D (\vec{\nabla} \cdot \vec{F}) \, dV = \oiint_S \vec{F} \cdot d\vec{A}.$$

Stokes' Theorem

Let S be a surface with the closed curve C as the edge of S. Orient the area element vectors $d\vec{A}$ and the curve C to have a right-hand relation. If $\vec{\nabla} \times \vec{F} = \vec{G}$, then

$$\iint\limits_S \vec{G} \cdot d\vec{A} = \oint\limits_C \vec{F} \cdot d\vec{r}.$$

By substituting, we can also write the conclusion as

$$\iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint\limits_C \vec{F} \cdot d\vec{r}.$$

Green's Theorem (as a special case of Stokes' Theorem)

Start with $\vec{F} = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath} + 0\,\hat{k}$.

$$\vec{\nabla} \times \vec{F} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\hat{k} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\hat{k}.$$

D: region in the xy-plane with closed curve C as edge.
Orient curve C counterclockwise.
Express area element vectors as $d\vec{A} = dx\,dy\,\hat{k}$.

$$(\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\hat{k} \cdot dx\,dy\,\hat{k} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dx\,dy.$$

Stokes' Theorem for this case:

$$\iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dx\,dy = \oint_C \big(P\,\hat{\imath} + Q\,\hat{\jmath}\big) \cdot d\vec{r}.$$

Green's Theorem (alternate notation)

Using an alternate notation for line integrals, Green's Theorem can also be written as

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \oint_C P\, dx + Q\, dy.$$

All together now

FTC	$\int_a^b F'(x)\,dx = F(b) - F(a)$
FTC for line integrals	$\int\limits_C \vec{\nabla} V \cdot d\vec{r} = V(B) - V(A)$
Divergence	$\iiint\limits_D (\vec{\nabla} \cdot \vec{F})\,dV = \oiint\limits_S \vec{F} \cdot d\vec{A}$
Stokes'	$\iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint\limits_C \vec{F} \cdot d\vec{r}$
Green's	$\iint\limits_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dx\,dy = \oint\limits_C P\,dx + Q\,dy.$

Common structure

The theorems given above all have the same all of which have the same basic structure:

Integrating the derivative of a function over a region gives the same value as integrating the function itself over the edge of the region.

In the case of a one-dimensional region such as a curve, the edge consists of only two points so integrating over the edge reduces to simply adding together two values.

FTC $$\int_a^b F'(x)\,dx = F(b) - F(a) = (-1)F(a) + F(b)$$

Factor of -1 accounts for orientation: at $x = a$, the direction pointing out of the segment $[a, b]$ is the negative direction while at $x = b$, the outward pointing direction is the positive direction.

Importance/utility of the fundamental theorems

Aesthetics: Beautiful unity among the various types of function, derivative, and integral we have explored in calculus.

Utility: Rather than evaluate an integral directly, we can trade it in for a related expression that is easier to evaluate. You are very familiar with doing this when you trade in a definite integral $\int_a^b f(x)\,dx$ for the sum $(-1)F(a) + F(b) = F(b) - F(a)$.

Utility: Given information about the derivative of a function at each point in a region, we can deduce information about certain integrals for the function itself (and vice versa).