Divergence

- Define: *divergence* of $\vec{F}$ at $\mathcal{P}$ is the *flux density* at $\mathcal{P}$:

$$\operatorname{div} \vec{F}(\mathcal{P}) = \lim_{\text{“}\Delta D \to \mathcal{P}\text{''}} \frac{\displaystyle\oiint_{\Delta S} \vec{F} \cdot d\vec{A}}{\Delta V}$$

- Compute: For $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}$, have

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \vec{\nabla} \cdot \vec{F}.$$

- Interpret:
  If $\vec{F}$ is fluid flow velocity, can think of $\operatorname{div} \vec{F}$ in the following way: for a given point (x, y, z), the number $\operatorname{div} \vec{F}(x, y, z)$ gives the rate at which fluid is being injected into the flow at that point.

Divergence Theorem

$$\iiint_D \vec{\nabla} \cdot \vec{F}\, dV = \iiint_D \frac{\displaystyle\oiint_{\Delta S} \vec{F} \cdot d\vec{A}}{dV}\, dV$$

$$= \iiint_D \frac{\text{flux through surface of infinitesimal piece of } D}{dV}\, dV$$

$$= \iiint_D \text{flux through surface of infinitesimal piece of } D$$

$$= \text{flux through surface of } D$$

$$= \oiint_S \vec{F} \cdot d\vec{A}$$

Curl

- Define: $\hat{n}$-component of *curl* of $\vec{F}$ at $\mathcal{P}$ is the *circulation density* at $\mathcal{P}$:

$$(\text{curl}\,\vec{F}) \cdot \hat{n} = \lim_{\text{“}\Delta C \to \mathcal{P}\text{''}} \frac{\oint_{\Delta C} \vec{F} \cdot d\vec{r}}{\Delta A}$$

- Compute: For $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}$, have

$$\text{curl}\,\vec{F} = \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\,\hat{\imath} + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\,\hat{\jmath} + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,\hat{k} = \vec{\nabla} \times \vec{F}.$$

- Interpret:
If $\vec{F}$ is fluid flow velocity, can think of curl $\vec{F}$ in the following way: for a given point (x, y, z), the direction of the vector curl $\vec{F}(x, y, z)$ is the direction in which to orient an infinitesimal paddlewheel to get the fastest rotation rate and the magnitude of curl $\vec{F}(x, y, z)$ is proportional to that rotation rate.

Stokes' Theorem

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dA$$

$$= \iint_S \frac{\oint_{\Delta C} \vec{F} \cdot d\vec{r}}{dA}\, dA$$

$$= \iint_S \frac{\text{circulation around edge of infinitesimal piece of } S}{dA}\, dA$$

$$= \iint_S \text{circulation around edge of infinitesimal piece of } S$$

$$= \text{circulation around edge of } S$$

$$= \oint_C \vec{F} \cdot d\vec{r}$$