Curl of a vector field

Circulation

Given a vector field $\vec{F}$ and an oriented closed loop C in space, we can think of the line integral $\oint_C \vec{F} \cdot d\vec{r}$ as a *circulation*. In this interpretation, we think of $\vec{F}$ as the velocity field of a fluid flow and think about the curve C as a rigid wire in this flow. Now think about pushing a bead that is strung on the wire. The fluid flow may help or hinder as we push the bead around the entire loop. The line integral sums up the tangential component of the velocity multiplied by displacement along the curve. At a point where the tangential component of the velocity is in the same direction as the displacement, the contribution to the line integral is positive. This corresponds to the fluid flow helping us in pushing the bead. At a point where the tangential component of the velocity is in the opposite direction to the displacement, the contribution to the line integral is negative. This corresponds to the fluid flow hindering us in pushing the bead. The line integral gives the net help along the entire curve. Thus, circulation is a measure of how much the flow helps if we push a bead around the wire in the direction specified by the orientation. If the circulation is positive, the fluid flow is a net help. If the circulation is negative, the fluid flow is a net hinderance.

Curl as circulation density

Start with a vector field $\vec{F}$ and focus on a point $\mathcal{P}$ in the domain of the vector field. Imagine a small, flat region that contains $\mathcal{P}$. (You can think of a rectangle or disk if it helps to be specific about the shape.) We will use ΔD to denote this planar region. Here Δ doesn't mean "a small change in" but serves to remind us that the region is small. Let $\Delta\vec{A}$ be the area vector for ΔD. Let ΔC be the closed curve that is the edge of this region. Orient the curve ΔC so that the thumb of your right hand points in the direction of $\Delta\vec{A}$ when your fingers curl around in the direction of ΔC as shown in Figure 1.

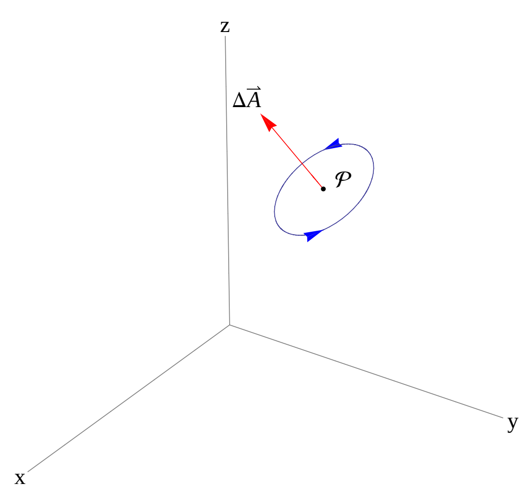

Figure 1. The edge ΔC of a small planar region.

Both the circulation $\oint_{\Delta C} \vec{F} \cdot d\vec{r}$ and the area ΔA will go to zero as we shrink the curve ΔC down to the point $\mathcal{P}$. However, the limit of the ratio

$$\frac{\displaystyle\oint_{\Delta C} \vec{F} \cdot d\vec{r}}{\Delta A}$$

might exist. If so, this limit is the *circulation density*. That is, the limit of the ratio is the circulation per unit area. We define the *curl of the vector field* $\vec{F}$ in terms of this circulation density. The curl of $\vec{F}$ is itself a vector field. The following definition gives the $\hat{n}$-component of the curl of $\vec{F}$ as a circulation density.

The $\hat{n}$-component of the curl of $\vec{F}$ at $\mathcal{P}$ is defined as the circulation density at a point $\mathcal{P}$ for a loop with $\Delta\vec{A} = \Delta A\,\hat{n}$. That is,

$$(\operatorname{curl} \vec{F}) \cdot \hat{n} = \lim_{\text{“}\Delta C \to \mathcal{P}\text{''}} \frac{\displaystyle\oint_{\Delta C} \vec{F} \cdot d\vec{r}}{\Delta A} \tag{1}$$

In a fluid flow interpretation, we can think of circulation density as telling us about infinitesimal vortices or “whirlpools” in the fluid flow. One way to conceptualize this is to think of a small paddlewheel anchored at a point in the fluid flow. The paddlewheel has an orientation given by the direction of the central axis which we will label $\hat{p}$. The blades of the paddlewheel are perpendicular to the central axis. Having nonzero circulation around an infinitesimal loop corresponds to an infinitesimal vortex in the fluid flow which will cause the paddlewheel to rotate about its axis. The component of curl $\vec{F}$ in the direction of $\hat{p}$ is proportional to the rotation rate of the paddlewheel. Positive rotation is in the counterclockwise direction when looking straight down theaxis of the paddlewheel. This is equivalent to saying the positive rotation is the direction in which the fingers of your right hand curl if you orient your thumb in the direction $\hat{p}$.

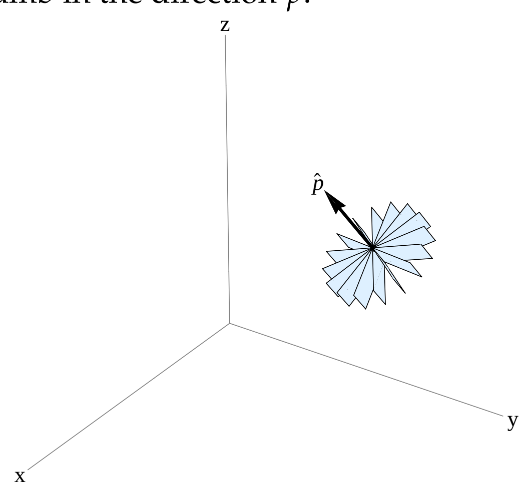

Figure 2. A paddlewheel with orientation $\hat{p}$

Example 1
Compute the $\hat{k}$-component of the curl of the vector field $\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath} + 0\,\hat{k}$ at the origin $(0,0,0)$.

Since we want the $\hat{k}$-component of the curl, we will choose a closed loop in the xy-plane so $\Delta\vec{A}$ is in the $\hat{k}$ direction. This vector field points tangent to any circle centered on the z-axis so a convenient choice of ΔC is a circle of radius R oriented with area vector pointing in the $\hat{k}$ direction. The area enclosed by the circle is $\Delta A = \pi R^2$. The circulation for this vector field around the circle of radius R is easy to compute as

$$\oint_{\text{circle}} \vec{F}\cdot d\vec{r} = \|\vec{F}\|(\text{circumference of the circle} = R(2\pi R) = 2\pi R^2.$$

So, we can form the ratio

$$\frac{\displaystyle\oint_{\Delta C} \vec{F}\cdot d\vec{r}}{\Delta A} = \frac{2\pi R^2}{\pi R^2} = 2.$$

We thus have

$$(\operatorname{curl}\vec{F})\cdot\hat{k} = \lim_{\text{“}\Delta C\to\mathcal{P}\text{''}} \frac{\displaystyle\oint_{\Delta C} \vec{F}\cdot d\vec{r}}{\Delta A} = \lim_{R\to 0} 2 = 2.$$

So, the circulation density for the orientation $\hat{k}$ for $\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath} + 0\,\hat{k}$ at the origin is positive. In a fluid flow interpretation, we can think of this as saying a paddle-wheel at $(0,0,0)$ with orientation $\hat{p} = \hat{k}$ is rotated in the positive direction by the flow of the fluid.

An expression for curl in cartesian coordinates

From the definition of curl in terms of circulation density, we learn what curl tells us about the vector field. However, computing the curl from this definition is difficult. We'll next look at getting an expression for the curl in terms of partial derivatives with respect to cartesian coordinates.

Let the vector field $\vec{F}$ be given in cartesian coordinates by

$$\vec{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}.$$

The curl of $\vec{F}$ is itself a vector field so we are looking for an expression of the form

$$\operatorname{curl}\vec{F} = (\hat{\imath}\text{-component})\,\hat{\imath} + (\hat{\jmath}\text{-component})\,\hat{\jmath} + (\hat{k}\text{-component})\,\hat{k}.$$

We'll look at the $\hat{k}$-component in detail and leave the others as exercises. Let ΔD be a small rectangle parallel to the xy-plane with $\Delta\vec{A} = \Delta A\hat{k}$. Then ΔC is oriented counter-clockwise as viewed from above. (See Figure 3.) Let $\mathcal{P}$ be one corner of this rectangle. Let Δx and Δy be the side lengths parallel to the x-axis and y-axis respectively. The area of the rectangle is $\Delta A = \Delta x\,\Delta y$.

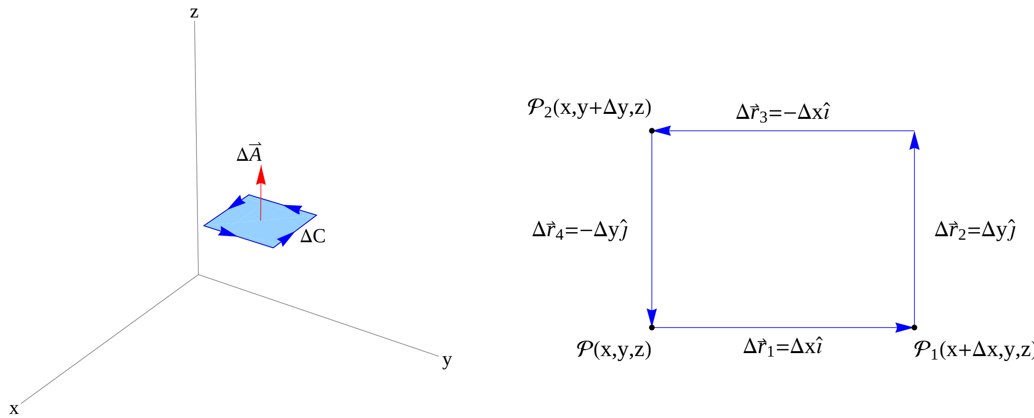

Figure 2. The small rectanglular loop ΔC shown in perspective (left) and from above (right).

Now let's think about the circulation for this rectangle as given by the line integral $\oint_{\Delta C} \vec{F} \cdot d\vec{r}$. Since we will be eventually taking a limit as the rectangle shrinks to the point $\mathcal{P}$, we can approximate this line integral with just four terms, one for each side of the rectangle. Let $\Delta\vec{r}_1$, $\Delta\vec{r}_2$, $\Delta\vec{r}_3$, and $\Delta\vec{r}_4$ be the displacement vectors along the four sides as shown in Figure 3. For each side, we can choose where to evaluate the vector field $\vec{F}$. We will use the corners $\mathcal{P}$, $\mathcal{P}_1$, and $\mathcal{P}_2$ shown in Figure 3. Specifically, we use

$$\oint_{\Delta C} \vec{F} \cdot d\vec{r} \approx \vec{F}(\mathcal{P}) \cdot \Delta\vec{r}_1 + \vec{F}(\mathcal{P}_1) \cdot \Delta\vec{r}_2 + \vec{F}(\mathcal{P}_2) \cdot \Delta\vec{r}_3 + \vec{F}(\mathcal{P}) \cdot \Delta\vec{r}_4. \tag{2}$$

Now let's introduce coordinates and components for the vectors in Equation (2). The vector field $\vec{F}$ has components $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}$. We choose coordinates with $\mathcal{P}(x, y, z)$. Since the side lengths are Δx and Δy, this gives us coordinates $\mathcal{P}_1(x + \Delta x, y, z)$ and $\mathcal{P}_2(x, y + \Delta y, z)$. From the geometry, we can see that $\Delta\vec{r}_1 = \Delta x\,\hat{\imath}$. Thus

$$\vec{F}(\mathcal{P}) \cdot \Delta\vec{r}_1 = \Big(P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{k}\Big) \cdot \big(\Delta x\,\hat{\imath}\big) = P(x,y,z)\Delta x.$$

The other three terms are similar:

$$\begin{aligned}
\Delta\vec{r}_2 &= \Delta y\,\hat{\jmath} \quad \text{so} \quad \vec{F}(\mathcal{P}_1) \cdot \Delta\vec{r}_2 = Q(x, y + \Delta x, z)\Delta y \\
\Delta\vec{r}_3 &= -\Delta x\,\hat{\imath} \quad \text{so} \quad \vec{F}(\mathcal{P}_2) \cdot \Delta\vec{r}_3 = -P(x, y + \Delta y, z)\Delta x \\
\Delta\vec{r}_4 &= -\Delta y\,\hat{\jmath} \quad \text{so} \quad \vec{F}(\mathcal{P}) \cdot \Delta\vec{r}_4 = -Q(x, y, z)\Delta y
\end{aligned}$$

Stop here and make sure you understand how to get these expressions including the negative signs in the last two of these.

Now substitute into Equation (2) to get

$$\begin{aligned}\oint_{\Delta C} \vec{F}\cdot d\vec{r} &\approx P(x,y,z)\Delta x + Q(x,y+\Delta x,z)\Delta y - P(x,y+\Delta y,z)\Delta x - Q(x,y,z)\Delta y\\ &= \big[P(x,y,z) - P(x,y+\Delta y,z)\big]\Delta x + \big[Q(x,y+\Delta x,z) - Q(x,y,z)\big]\Delta y\\ &= \big[Q(x,y+\Delta x,z) - Q(x,y,z)\big]\Delta y - \big[P(x,y+\Delta y,z) - P(x,y,z)\big]\Delta x\end{aligned}$$

Thus, the ratio of circulation to area enclosed is

$$\begin{aligned}\frac{\oint_{\Delta C}\vec{F}\cdot d\vec{r}}{\Delta A} &= \frac{\big[Q(x,y+\Delta x,z) - Q(x,y,z)\big]\Delta y - \big[P(x,y+\Delta y,z) - P(x,y,z)\big]\Delta x}{\Delta x\,\Delta y}\\ &= \frac{Q(x,y+\Delta x,z) - Q(x,y,z)}{\Delta x} - \frac{P(x,y+\Delta y,z) - P(x,y,z)}{\Delta y}. \qquad (3)\end{aligned}$$

Now consider the limit as ΔC shrinks to the point $\mathcal{P}$. We achieve this by taking $\Delta x \to 0$ and $\Delta y \to 0$. Note each of the two terms on the right side of Equation (3) will give a partial derivative in this limit. We thus have

$$\hat{k}\text{-component of curl}\,\vec{F} \text{ at } \mathcal{P} \;=\; \lim_{\Delta x,\Delta y\to 0} \frac{\oint_{\Delta C}\vec{F}\cdot d\vec{r}}{\Delta A} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \qquad (4)$$

for an "infinitesimal loop" parallel to the xy-plane. Another way to specify this orientation is to say the area vector for this "infinitesimal loop" is $d\vec{A} = dA\,\hat{k}$.

In the problems, you are asked to compute the $\hat{\imath}$- and $\hat{\jmath}$-components of the curl of $\vec{F}$. The answers are

$$\hat{\imath}\text{-component of curl}\,\vec{F} \text{ at } \mathcal{P} \;=\; \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \qquad (5)$$

$$\hat{\jmath}\text{-component of curl}\,\vec{F} \text{ at } \mathcal{P} \;=\; \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}. \qquad (6)$$

Putting these together, we have the following result.

In cartesian coordinates, the curl of

$$\vec{F} = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$$

is given by

$$\operatorname{curl}\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}. \qquad (7)$$

Example 2

Compute the curl of $\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath} + 0\,\hat{k}$.

We can read off the components as $P(x,y,z) = -y$, $Q(x,y,z) = x$, and $R(x,y,z) = 0$. So, we can compute the curl components as

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = \frac{\partial}{\partial y}[0] - \frac{\partial}{\partial z}[x] = 0 - 0 = 0$$
$$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = \frac{\partial}{\partial z}[-y] - \frac{\partial}{\partial z}[0] = 0 - 0 = 0$$
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}[x] - \frac{\partial}{\partial y}[-y] = 1 - (-1) = 2.$$

Thus, the curl of this vector field is

$$\operatorname{curl} \vec{F}(x,y,z) = 0\,\hat{\imath} + 0\,\hat{\jmath} + 2\,\hat{k}$$

for all points (x,y,z). In Example 1, we computed the value of 2 for the $\hat{k}$ component of this vector field at $(0,0,0)$. Now we see that this vector field has a constant curl with zero $\hat{\imath}$ and $\hat{\jmath}$ components and 2 for the $\hat{k}$ component. In a fluid flow interpretation, a paddlewheel placed at any point oriented with $\hat{p}$ as $\hat{\imath}$ or $\hat{\jmath}$ will not be rotated by the flow while a paddlewheel at any point oriented with $\hat{p} = \hat{k}$ will be rotated in the positive direction.

The operator point of view

We have defined the "del" operator as, in cartesian coordinates,

$$\vec{\nabla} = \hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}.$$

We have seen that the divergence of a vector field is given by dotting $\vec{\nabla}$ with $\vec{F}$. For fun, let's see what happens if we *cross* $\vec{\nabla}$ with $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}$:

$$\begin{aligned}
\vec{\nabla} \times \vec{F} &= \Big(\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\Big) \times \Big(P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}\Big) \\
&= \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\,\hat{\imath} - \Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big)\,\hat{\jmath} + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,\hat{k} \\
&= \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\,\hat{\imath} + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\,\hat{\jmath} + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,\hat{k} \\
&= \operatorname{curl} \vec{F}.
\end{aligned}$$

So, we have a convenient way of remembering how to compute the curl of a vector field. We need only cross $\vec{\nabla}$ with $\vec{F}$.

Problems: curl of a vector field

1. Compute the $\hat{\jmath}$ and $\hat{k}$ components of the curl in cartesian coordinates by suitably modifying the plan carried out for the $\hat{\imath}$ component.

For each of the following vector fields, compute the curl. Evaluate the curl at a few points and give an interpretation for each value.

2. $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + 0\,\hat{k}$

3. $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$

4. $\vec{F} = z\sin(xy)\,\hat{\imath} + (x+y)\,\hat{\jmath} + ze^{x}\,\hat{k}$

5. $\vec{F} = \dfrac{-y\,\hat{\imath} + x\,\hat{\jmath}}{\sqrt{x^2+y^2}}$

6. $\vec{F} = \dfrac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2+y^2}$

7. Consider a planar vector field $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$. We can think of this as a vector field in space by adding $0\,\hat{k}$ to get $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath} + 0\,\hat{k}$. Compute the curl of a vector field with this form.

8. Let $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}$ be a vector field with the component functions P, Q, and R having continuous second partial derivatives. Show that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$.

9. Let f be a function with continuous second partial derivatives. Show that $\vec{\nabla} \times \vec{\nabla} f = \vec{0}$.