Fundamental theorems of calculus

Note: In each of the following theorems, hypotheses on continuity of the integrand and "niceness" of the relevant region are omitted in order to focus on other details.

Fundamental Theorem for Definite Integrals

If $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.
By substituting, we can also write the conclusion as

$$\boxed{\int_a^b F'(x)\,dx = F(b) - F(a).}$$

Fundamental Theorem for Line Integrals

Let C be a curve that starts at A and ends at B. If $\vec{\nabla} V = \vec{F}$, then

$$\int\limits_C \vec{F} \cdot d\vec{r} = V(B) - V(A).$$

By substituting, we can also write the conclusion as

$$\boxed{\int\limits_C \vec{\nabla} V \cdot d\vec{r} = V(B) - V(A).}$$

Divergence Theorem

Let D be a solid region with the closed surface S as the edge of D and area element vectors $d\vec{A}$ for S oriented outward. If $\vec{\nabla} \cdot \vec{F} = f$, then

$$\iiint\limits_D f\,dV = \oiint\limits_S \vec{F} \cdot d\vec{A}.$$

By substituting, we can also write the conclusion as

$$\boxed{\iiint\limits_D (\vec{\nabla} \cdot \vec{F})\,dV = \oiint\limits_S \vec{F} \cdot d\vec{A}.}$$

Stokes' Theorem

Let S be a surface with the closed curve C as the edge of S. Orient the area element vectors $d\vec{A}$ and the curve C to have a right-hand relation. If $\vec{\nabla} \times \vec{F} = \vec{G}$, then

$$\iint\limits_S \vec{G} \cdot d\vec{A} = \oint\limits_C \vec{F} \cdot d\vec{r}.$$

By substituting, we can also write the conclusion as

$$\boxed{\iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint\limits_C \vec{F} \cdot d\vec{r}.}$$

Green's Theorem
We can derive Green's Theorem as a special case of Stokes' Theorem. Consider a vector field of the form $\vec{F} = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath} + 0\,\hat{k}$. Note that the curl of $\vec{F}$ is

$$\vec{\nabla} \times \vec{F} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\hat{k} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\hat{k}.$$

Let D be a planar region in the xy-plane with the closed curve C as the edge of D. Orient the curve C counterclockwise. If we think of D as a surface, we can express the area element vectors as $d\vec{A} = dx\,dy\,\hat{k}$.
We now compute

$$(\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\hat{k} \cdot dx\,dy\,\hat{k} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dx\,dy.$$

Using this special case in the conclusion of Stokes' Theorem, we get

$$\iint\limits_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dx\,dy = \oint\limits_C (P\,\hat{\imath} + Q\,\hat{\jmath}) \cdot d\vec{r}.$$

Using an alternate notation for line integrals, this can also be written as

$$\boxed{\iint\limits_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dx\,dy = \oint\limits_C P\,dx + Q\,dy.}$$

Common structure among these fundamental theorems
The theorems given above all have the same all of which have the same basic structure: *Integrating the derivative of a function over a region gives the same value as integrating the function itself over the edge of the region*. In the case of a one-dimensional region such as a curve, the edge consists of only two points so integrating over the edge reduces to simply adding together two values. Here's how this basic idea plays out in the specific cases:

- In the Fundamental Theorem for Definite Integrals, the region is an interval $[a,b]$ on the input axis so the edge of the region consists of two points a and b on the axis. The function is a function of one variable and the derivative is the first kind of derivative you learned about. In words, the theorem says that integrating the derivative F' over the interval $[a,b]$ is the same as adding up the function F for the two endpoints. But wait, isn't $F(b) - F(a)$ a difference rather than a sum? Yes, but we can think of it as $(-1)F(a) + F(b)$. The factor of -1 relates to the issue of orientation. At a, the direction pointing out is the negative direction while at b, the outward pointing direction is the positive direction. The factor of -1 reflects the fact that the outward direction at a is the negative direction.
- In the Fundamental Theorem for Line Integrals, the region is a curve C so the edge consists of two points A and B or on the plane or in space. The function is a function of two or more variable and the derivative is the gradient. In

words, the theorem says that integrating the gradient $\vec{\nabla} V$ over the curve C is the same as adding up the function V for the two endpoints. We usually write this as $V(B) - V(A)$ but can think of it as $(-1)V(a) + V(b)$. As above, the factor of -1 relates to the issue of orientation and is related to the fact that $d\vec{r}$ points into the curve at A and out of the curve at B.

- In Green's Theorem, the region is a planar region D with edge consisting of a closed curve C. The function is a planar vector field and the derivative is the $\hat{k}$ component of the curl (which is the only non-zero component of the curl for a planar vector field). In words, the theorem says that integrating the curl $\partial Q/\partial x - \partial P/\partial y$ over the region D is the same as integrating the vector field $P\,\hat{\imath} + Q\,\hat{\jmath}$ over the curve C.
- In Stoke's Theorem, the region is a surface S in space with edge consisting of a closed curve C. The function is a vector field and the derivative is the curl. In words, the theorem says that integrating the curl $\vec{\nabla} \times \vec{F}$ over the surface S is the same as integrating the vector field $\vec{F}$ over the curve C.
- In the Divergence Theorem, the region is a solid region in space with edge consisting of a closed surface S. The function is a vector field and the derivative is the divergence. In words, the theorem says that integrating the divergence $\vec{\nabla} \cdot \vec{F}$ over the solid region D is the same as integrating the vector field $\vec{F}$ over the surface S.

We can also organize these in terms of the dimension of the region and its edge:

- In the Fundmental Theorems for Definite Integrals and Line Integrals, the region is one-dimensional (an interval or a curve) and the edge is zero-dimensional (a set of two points).
- In Green's Theorem and Stoke's Theorem, the region is two-dimensional (a planar region or a surface) and the edge is one-dimensional (a curve).
- In the Divergence Theorem, the region is three-dimensional (a solid region) and the edge is two-dimensional (a surface).

Importance of the fundamental theorems

The fundamental theorems are important for both aesthetic value and as useful tools. Aesthetically, the fundamental theorems provide a beautiful unity among the various types of function, derivative, and integral we have explored in calculus. As tools, we use the fundamental theorems in two primary ways:

- Rather than evaluate an integral directly, we can trade it in for a related expression that is easier to evaluate. You are very familiar with doing this when you trade in a definite integral $\int_a^b f(x)\,dx$ for the sum $(-1)F(a) + F(b) = F(b) - F(a)$. Problems 1, 3, and 4 give you practice with this type of "trading in" using the other fundamental theorems.
- Given information about the derivative of a function at each point in a region, we can deduce information about certain integrals for the function itself (and vice versa). Problem 2 gives you an example of this use.

Problems: Fundamental theorems of calculus

1. Use the Divergence Theorem to evaluate $\oiint_S \vec{F} \cdot d\vec{A}$ where

$$\vec{F} = (z - x)\,\hat{\imath} + (x - y)\,\hat{\jmath} + (y - z)\,\hat{k}$$

   and S is the sphere of radius 4 centered at the origin with $d\vec{A}$ oriented outward.

   Answer: $\oiint_S \vec{F} \cdot d\vec{A} = -256\pi$

2. Suppose that $\vec{F}$ is a vector field with $\vec{\nabla} \cdot \vec{F} = 0$ for all points in $\mathbb{R}^3$. Show that $\oiint_S \vec{F} \cdot d\vec{A} = 0$ for any closed surface S in $\mathbb{R}^3$.

3. Use Stokes' Theorem (or Green's Theorem) to evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = y^2\,\hat{\imath} - x^2\,\hat{\jmath}$ and C is the square in the xy-plane with corners at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ traversed counterclockwise.

   Answer: $\oint_C \vec{F} \cdot d\vec{r} = -2$

4. Suppose C is a simple closed curve in the xy-plane. Let $\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath}$ and consider the line integral $\oint_C \vec{F} \cdot d\vec{r}$. Use Stokes' Theorem (or Green's Theorem) to relate the value of this line integral to the area of the region enclosed by C. Note: A *simple* curve is one with no self-intersections so a simple closed curve is a loop with no self-interesections.

5. Use your result from Problem 4 to compute the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.