## Conservative vector fields

Let $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}$ be a vector field that is continuous on some domain in $\mathbb{R}^3$. Let $D$ be an open, connected, and simply connected region in the domain of $\vec{F}$. The following conditions are equivalent (that is, any one of the conditions implies any of the other conditions):

1. There is a function $V$ such that $\vec{\nabla} V = \vec{F}$ for all point in $D$.

2. The following equations hold for all points in D:

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \text{and} \qquad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

   These equations are equivalent to the condition

$$\vec{\nabla} \times \vec{F} = \vec{0}.$$

3. $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for every pair of smooth curves in $C_1$ and $C_2$ in $D$ that have a common start point and a common end point.

4. $\displaystyle\oint_C \vec{F} \cdot d\vec{r} = 0$ for every smooth closed curve $C$ in $D$

A vector field satisfying any one (and hence all) of these properties is said to be *conservative* for the region $D$. A function $V$ satisfying the first property with respect to $\vec{F}$ is called a *potential function* for the vector field $\vec{F}$. The second property is called the *component test*. The third property is referred to as *path-independance*. The fourth property is sometimes called the *closed-loop* property.

The four properties are also equivalent for a vector field $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath}$ with domain in $\mathbb{R}^2$. In this case, the component test is simply

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$