A few problems on Gaussians

In class, we examined the heat equation solution

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}.$$

One way to think about this function is to view it as a Gaussian function. A Gaussian function has the form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

with $\sigma > 0$. In probability, this is also referred to as a *Gaussian distribution* or a *normal distribution*. A graph of this function is the *bell curve*. The heat equation solution is a Gaussian with $\sigma = \sqrt{2kt}$.

In class, we stated two basic properties of Gaussian functions that are useful in understanding their nature.

1. Prove that the Gaussian function f given above has inflection points at $x = \pm\sigma$.

2. Prove that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$ for all $\sigma > 0$ by verifying or filling in the details of the following steps. That is, you should verify the reasoning and calculations in each step (including justifying each equality in strings of equalities) and you should fill in any omitted computational details.

   (a) For convenience, let $I = \displaystyle\int_{-\infty}^{\infty} f(x)\,dx$. Note that the value of I does not care about the name of the integration variable so we can also write $I = \displaystyle\int_{-\infty}^{\infty} f(y)\,dy$.

   (b) By symmetry, $\frac{1}{2}I = \displaystyle\int_0^{\infty} f(x)\,dx = \int_0^{\infty} f(y)\,dy$.

   (c) So, $\frac{1}{4}I^2 = (\frac{1}{2}I)(\frac{1}{2}I) = \displaystyle\int_0^{\infty} f(x)\,dx \cdot \int_0^{\infty} f(y)\,dy = \int_0^{\infty}\int_0^{\infty} f(x)f(y)\,dxdy$.

   (d) Substituting specific expressions for $f(x)$ and $f(y)$ and doing some rearranging, we have

   $$\tfrac{1}{4}I^2 = \frac{1}{2\pi\sigma^2}\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)/2\sigma^2}\,dxdy$$

   (e) The iterated integral in (d) corresponds to a double integral over the first quadrant of the xy-plane. Transforming to polar coordinates, we can express this as

   $$\tfrac{1}{4}I^2 = \frac{1}{2\pi\sigma^2}\int_0^{\pi/2}\int_0^{\infty} e^{-r^2/2\sigma^2}\,rdrd\theta = \frac{1}{2\pi\sigma^2}\int_0^{\pi/2} d\theta \cdot \int_0^{\infty} e^{-r^2/2\sigma^2}\,rdr$$

   (f) Evaluating the θ- and r-integrals (using a substitution for the r integral), we get

   $$\tfrac{1}{4}I^2 = \frac{1}{2\pi\sigma^2}\cdot\frac{\pi}{2}\cdot\sigma^2 = \frac{1}{4}$$

   (g) Thus, $I = \int_{-\infty}^{\infty} f(x)\,dx = 1$.