Constructing definite integrals

1. Consider the problem of computing the total mass of a column of air. The density of air decreases as height above sea level increases. Let h be height above sea level measured in meters (m). Let ρ be the density of air, measured in kilograms per cubic meter (kg/m^3). Note that ρ varies with height h. (Here, ρ is the lower case Greek letter "rho".)

   (a) Construct a definite integral to compute the total mass of air in a cylindrical column of radius R and height H with its base at sea level.

   (b) Compute the total mass of air if $\rho(h) = \rho_0 e^{-kh}$ where ρ_0 and k are positive constants.

   (c) Get a numerical value for the total mass using the values $\rho_0 = 1.22$ kg/m^3, $k = 1.1 \times 10^{-4}$ m^{-1}, $R = 1$ m and $H = 10000$ m.

2. Charge is spread on a circle of radius R so that the density varies around the circle. (Note that circle here means the curve as opposed to a disk.) Let λ be the charge density measured in Coulombs per meter (C/m). Let θ measure the angle on the circle from a fixed reference ray (conventionally taken to be the positive x-axis). So, the charge density λ varies with angle θ.

   (a) Construct a definite integral to compute the total charge on the circle.

   (b) Compute the total charge if $\lambda(\theta) = \lambda_0(1 + \cos\theta)$ where λ_0 is a positive constant.

   (c) Get a numerical value for the total charge in (b) using the values $R = 0.25$ m and $\lambda_0 = 1.6 \times 10^{-3}$ C/m.

3. Consider the problem of computing the total number of bacteria in a circular petri dish. The bacteria colony is more dense at the center than at the edges of the petri dish. Let r denote radial distance from the center of the dish measured in centimeters (cm). Let σ be the density of the bacteria colony, measured in number per square centimeter (#/cm^2). Note that σ varies with radius r. (Here, σ is the lower case Greek letter "sigma".)

   (a) Construct a definite integral to compute the total number of bacteria in a petri dish of radius R.

   (b) Compute the total number of bacteria if the density is σ_0 at the center of the dish and decreases linearly to zero at the edge of the dish.

   (c) Get a numerical value for the total number with the density as in (b) and the values $\sigma_0 = 5.4 \times 10^3$ per cm^2 and $R = 5.5$ cm.

4. A hydrogen atom consists of one proton and one electron. A *free* hydrogen atom is one that experiences no external forces. In a free hydrogen atom, the electron can be in one of infinitely many discrete states. These states are labeled by three integers, usually denoted n, l, and m. For each state, there is an *electron location probability density* that gives the probability density (per volume) for the location of the electron as a function of position (measured with respect to the proton at the center of the atom). The states with $l = 0$ have probability densities that vary only with radial distance from the proton. (States with $l > 0$ have probability densities that also vary with angular directions.) Let r be the radial distance from the proton and let ρ be the electron probability density. So, probability density ρ varies with radial distance r.

   (a) Construct a definite integral to compute the total probability of finding an electron between radius $r = a$ and radius $r = b$.

   (b) The $n = 2$, $l = 0$, $m = 0$ state of a free hydrogen atom has an electron probability density given by

   $$\rho(r) = \frac{1}{32\pi}(2 - r)^2 e^{-r}.$$

   Here, the radial coordinate r is measured in units of *Bohr radii* where the Bohr radius is equal to about 5.3×10^{-11} meters. (So, for example, $r = 2$ means a radial distance of 2 Bohr radii or about 10.6×10^{-11} meters.)

   Compute the probability of an electron in the $n = 2$, $l = 0$, $m = 0$ state being between $r = 0$ and $r = 2$ Bohr radii.

   Note: For this and the following, it is sufficient to get a numerical estimate using technology.

   (c) Compute the probability of an electron in the $n = 2$, $l = 0$, $m = 0$ state being between $r = 2$ and $r = 4$ Bohr radii.

   (d) Compute the probability of an electron in the $n = 2$, $l = 0$, $m = 0$ state being between $r = 4$ and $r = 6$ Bohr radii.