

**Instructions:** You can work on the problems in any order. Please use just one side of each page and clearly number the problems. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem. (115 points total)

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1. For a function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , we can compute the gradient vector field  $\vec{\nabla} f$ .
  - (a) For a specific point  $P$ , what information do we get from the *direction* of  $\vec{\nabla} f(P)$ ? (3 points)
  - (b) For a specific point  $P$ , what information do we get from the *magnitude* of  $\vec{\nabla} f(P)$ ? (3 points)
  
2. (a) List two different types of integral that we have introduced in this course. For each type, give (4 points)
  - the type of function used as the integrand (in the form  $f : A \rightarrow B$ )
  - the type of geometric region or object over which the integration is done
 (b) What, fundamentally, is an integral? That is, what is common to all of types of integral that we have seen? (3 points)
  
3. Prove the following: If  $\vec{u}$  and  $\vec{v}$  are nonzero vectors, then  $|\vec{v}|\vec{u} + |\vec{u}|\vec{v}$  and  $|\vec{v}|\vec{u} - |\vec{u}|\vec{v}$  are perpendicular. (6 points)
  
4. The function  $\vec{r}(t) = 4t \sin(\pi t) \hat{i} + 4t \cos(\pi t) \hat{j} + 10t \hat{k}$  traces out a curve  $C$ .
  - (a) Find the value of  $t$  corresponding to the point  $(2, 0, 5)$  on  $C$ . (3 points)
  - (b) Find a vector tangent to the curve  $C$  at the point  $(2, 0, 5)$ . (6 points)
  - (c) Find an equation of the plane that contains the point  $(2, 0, 5)$  and that is perpendicular to the curve  $C$  at this point. (5 points)
  
5. Consider the function  $f(x, y) = \frac{x}{y^2}$ .
  - (a) Sketch a plot of the level curve that contains the point  $(8, 2)$ . (3 points)
  - (b) Compute the gradient of  $f$  for the point  $(8, 2)$ . Sketch this gradient vector on your plot from (a). (6 points)
  - (c) Compute the rate of change in  $f$  at  $(8, 2)$  in the direction of the point  $(4, 5)$ . (5 points)
  
6. Show that  $(-4, 2)$  is a critical input for the function  $f(x, y) = xy^4 + 2x^2 + 128y$  and classify this input as a local minimizer, a local maximizer, or neither. (12 points)

7. Consider the vector field  $\vec{F} = y\hat{i} + x\hat{j} + xyz\hat{k}$ .

- (a) Compute the divergence of  $\vec{F}$  for the point  $(2, 3, 1)$ . (4 points)
- (b) Give some interpretation of your result for (a). (3 points)
- (c) Compute the curl of  $\vec{F}$  for the point  $(2, 3, 1)$ . (4 points)
- (d) Give some interpretation of your result for (c). (3 points)

8. Consider the planar vector field  $\vec{F} = x^2\hat{j}$ .

- (a) Sketch a vector field plot of  $\vec{F}$  for the region given by  $0 \leq x \leq 2$  and  $0 \leq y \leq 2$ . (4 points)
- (b) Let  $C_1$  be the line segment from  $(0, 2)$  to  $(-2, 0)$ . Draw  $C_1$  on your vector field plot from (a). Determine if  $\int_{C_1} \vec{F} \cdot d\vec{r}$  is negative, zero, or positive. Give a geometric justification for your conclusion. (4 points)
- (c) Let  $C_2$  be the line segment from  $(0, 1)$  to  $(1, 1)$ . Draw  $C_2$  on your vector field plot from (a). Determine if  $\int_{C_2} \vec{F} \cdot d\vec{r}$  is negative, zero, or positive. Give a geometric justification for your conclusion. (4 points)

9. Do any three of the following four problems. Circle the problem label of the problems you are submitting. (10 points each)

(A) Consider the function  $f(x, y, z) = xz \cos(\pi y)$ . Compute  $\int_C \vec{\nabla} f \cdot d\vec{r}$  where  $C$  is any curve from  $(1, 0, 1)$  to  $(3, 2, 5)$

(B) Set up an iterated integral (or integrals) equal to the triple integral  $\iiint_D f dV$  where  $f(x, y, z) = x^2 z^2$  and  $D$  is the solid region inside both the cylinder  $x^2 + y^2 = 4$  and the sphere of radius 6 centered at the origin. Express any iterated integral entirely in terms of three variables (of your own choosing). You do not need to evaluate the integral(s).

(C) Charge is distributed on a hemisphere of radius  $R$ . Think of this as the northern hemisphere of the earth. The area charge density is proportional to the distance from the plane containing the equator with a value of 0 on the equator and a value of  $\sigma_0$  at the north pole. Compute the total charge on the hemisphere in terms of  $R$  and  $\sigma_0$ .

(D) Evaluate the surface integral  $\oiint_S \vec{F} \cdot d\vec{A}$  where  $\vec{F} = 3x\hat{i} + 2y\hat{j} + 5z\hat{k}$  and  $S$  is the closed surface that consists of the right circular cylinder  $x^2 + y^2 = 9$  for  $-1 \leq z \leq 4$  together with a disk at the bottom and a disk at the top.

Hint: Use the Divergence Theorem.