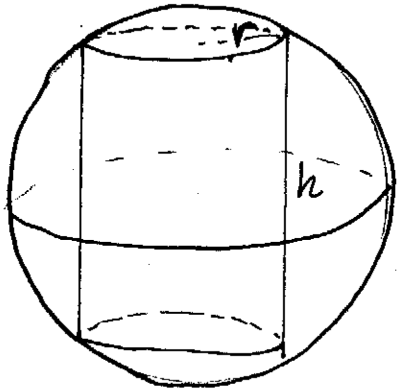
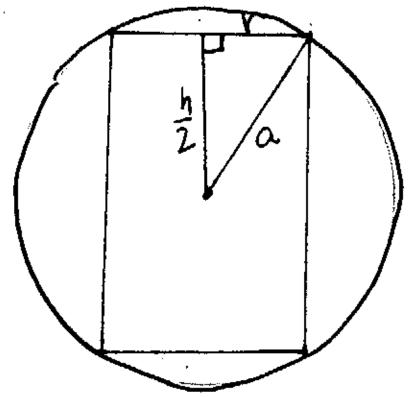
Section 12.8 #10

Approach #1 : no coordinates

perspective view:



cross-section



objective : maximize $A = 2\pi r h$

constraint : $r^2 + \frac{1}{4}h^2 = a^2$ (by Pythagorean thm in cross-section)

$$\vec{\nabla} A = 2\pi h \hat{r} + 2\pi r \hat{h} \qquad \vec{\nabla} g = 2r\hat{r} + \tfrac{1}{2}h\hat{h}$$

$$\vec{\nabla} A = \lambda \vec{\nabla} g \implies \begin{aligned} 2\pi h &= \lambda \cdot 2r \quad (1) \\ 2\pi r &= \lambda \cdot 2h \quad (2) \\ r^2 + \tfrac{1}{4}h^2 &= a^2 \quad (3) \end{aligned}$$

$$\left.\begin{aligned} (1) &\implies \lambda = \frac{\pi h}{r} \\ (2) &\implies \lambda = \frac{4\pi r}{h} \end{aligned}\right\} \implies \frac{\pi h}{r} = \frac{4\pi r}{h} \implies h^2 = 4r^2$$

Substitute into (3):

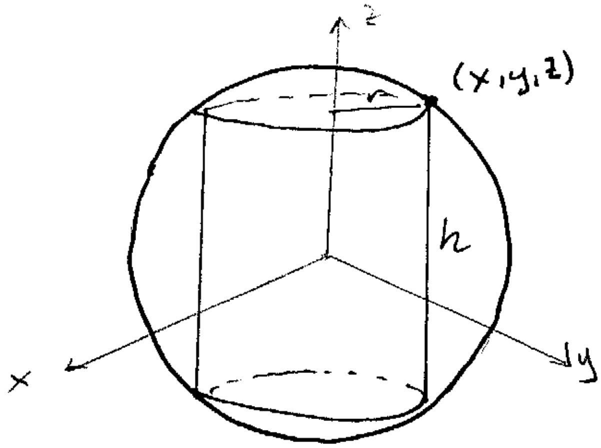
$$r^2 + \tfrac{1}{4}(4r^2) = a^2 \implies 2r^2 = a^2 \implies r = \frac{a}{\sqrt{2}}$$

$$\implies h = 2r = \sqrt{2}\, a$$

Maximum surface area :

$$A = 2\pi \cdot \frac{a}{\sqrt{2}} \cdot \sqrt{2}\, a = 2\pi a^2$$

Approach #2 : with coordinates



cylinder : $x^2+y^2=r^2$

sphere : $x^2+y^2+z^2=a^2$

$h = 2z \qquad r^2 = x^2+y^2$

where (x,y,z) is a point where cylinder & sphere intersect.

objective: maximize $A = 2\pi r h = 2\pi\sqrt{x^2+y^2}\cdot 2z = 4\pi z\sqrt{x^2+y^2}$

constraint: $x^2+y^2+z^2=a^2$

$$\vec{\nabla} A = 4\pi \frac{xz}{\sqrt{x^2+y^2}}\,\hat{\imath} + 4\pi\frac{yz}{\sqrt{x^2+y^2}}\,\hat{\jmath} + 4\pi\sqrt{x^2+y^2}\,\hat{k}$$

$$\vec{\nabla} g = 2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\hat{k}$$

$$\vec{\nabla} A = \lambda \vec{\nabla} g \quad \Rightarrow \quad 4\pi\frac{xz}{\sqrt{x^2+y^2}} = \lambda\cdot 2x \qquad (1)$$

$$4\pi\frac{yz}{\sqrt{x^2+y^2}} = \lambda\cdot 2y \qquad (2)$$

$$4\pi\sqrt{x^2+y^2} = \lambda\cdot 2z \qquad (3)$$

$$x^2+y^2+z^2=a^2 \qquad (4)$$

$$\left.\begin{array}{l}(1)\text{ or }(2) \Rightarrow \lambda = 2\pi\dfrac{z}{\sqrt{x^2+y^2}} \\ (3) \Rightarrow \lambda = 2\pi\dfrac{\sqrt{x^2+y^2}}{z}\end{array}\right\} \Rightarrow \frac{z}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2}}{z} \Rightarrow z^2 = x^2+y^2$$

Substitute z^2 for x^2+y^2 in (4):

$$z^2+z^2=a^2 \Rightarrow 2z^2=a^2 \Rightarrow z = \frac{a}{\sqrt{2}}$$

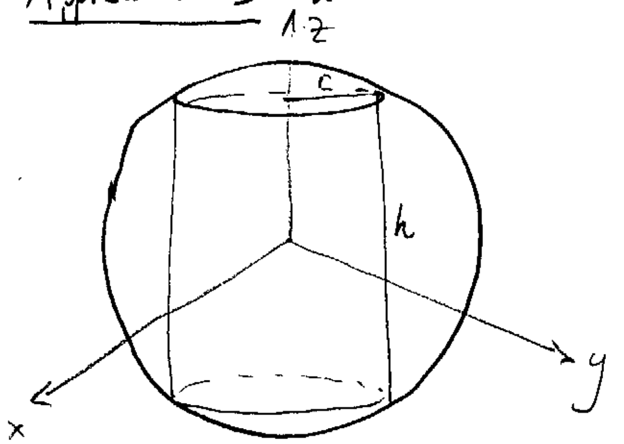
So $h = 2z = 2\dfrac{a}{\sqrt{2}} = \sqrt{2}\,a$

$r = \sqrt{x^2+y^2} = z = \dfrac{a}{\sqrt{2}}$

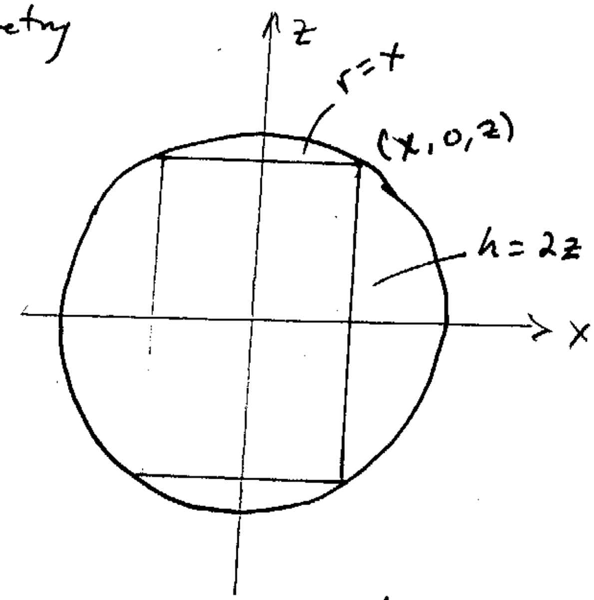
Maximum surface area:

$$A = 2\pi\frac{a}{\sqrt{2}}\cdot\sqrt{2}\,a = 2\pi a^2$$

Approach #3: with coordinates and symmetry



perspective



$y=0$ cross-section

cylinder: $x^2+y^2=r^2$

sphere: $x^2+y^2+z^2=a^2$

with $y=0$: $x^2=r^2 \implies x=r$

$x^2+z^2=a^2$

Objective: maximize $A = 2\pi r h = 2\pi x(2z) = 4\pi x z$

constraint: $x^2+z^2=a^2$

$\vec{\nabla} A = 4\pi z\,\hat{i} + 4\pi x\,\hat{j} \qquad \vec{\nabla} g = 2x\,\hat{i} + 2z\,\hat{j}$

$$\vec{\nabla} A = \lambda \vec{\nabla} g \implies \begin{aligned} 4\pi z &= \lambda \cdot 2x \quad (1) \\ 4\pi x &= \lambda \cdot 2z \quad (2) \\ x^2+z^2 &= a^2 \quad (3) \end{aligned}$$

$$\left.\begin{aligned} (1) &\implies \lambda = \frac{2\pi z}{x} \\ (2) &\implies \lambda = \frac{2\pi x}{z} \end{aligned}\right\} \quad \frac{2\pi z}{x} = \frac{2\pi x}{z} \implies z^2 = x^2$$

Substitute into (3): $x^2+x^2=a^2 \implies 2x^2=a^2 \implies x = a/\sqrt{2}$

$\implies z = a/\sqrt{2}$

So $r = x = a/\sqrt{2}$ and $h = 2z = 2\frac{a}{\sqrt{2}} = \sqrt{2}\,a$

Maximum surface area: $A = 2\pi r h = 2\pi \cdot \frac{a}{\sqrt{2}} \cdot \sqrt{2}\,a = 2\pi a^2$