Example: evaluating a double integral using polar coordinates

Problem: Compute the area enclosed by one petal of the rose curve $r = \cos(3\theta)$.

Solution: The petal curve is shown in the figure on the left below. To get the area, we will add up small contributions to the area over one petal region. So, we need to evaluate the double integral

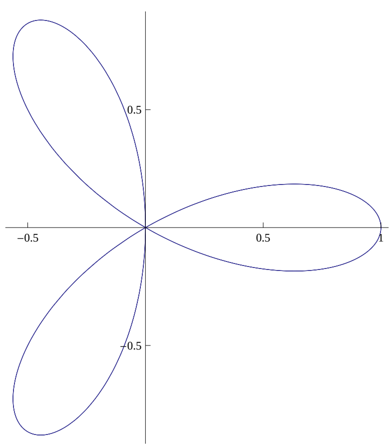
$$A = \iint_{\text{petal}} dA.$$

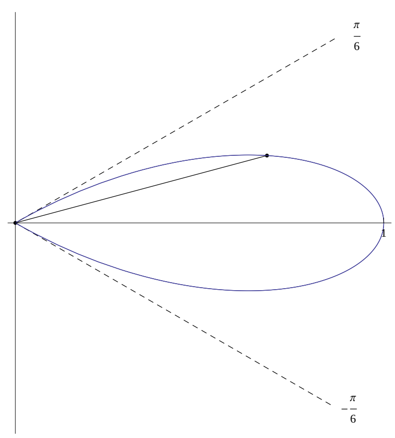
We will use polar coordinates so we need to take care of three things:

1. Describe the region with appropriate bounds on r and θ.
2. Find an expression for the integrand in terms of r and θ.
3. Use $dA = r\,dr\,d\theta$ or $dA = r\,d\theta\,dr$ with the choice dictated by which of the variables has constant bounds.

Step 1 For the region, we will focus on the petal bisected by the positive x-axis. This petal is contained in a certain wedge in the θ direction. The endpoint values for this wedge are determined by values of θ for which $r = 0$ along the petal curve. Since the curve is given by $r = \cos(3\theta)$, we need to think about solutions of $\cos(3\theta) = 0$. The first positive solution will be given by the condition $3\theta = \pi/2$ so $\theta = \pi/6$. By symmetry, we can then see that the first petal is contained in the wedge from $\theta = -\pi/6$ to $\theta = \pi/6$ as shown in the figure on the right below. If we consider a generic angle θ in this wedge, we see that r varies from 0 to $\cos(3\theta)$ in going across the petal region. So, we can describe the region with bounds on r and θ as

$$-\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \qquad \text{and} \qquad 0 \le r \le \cos(3\theta).$$





Step 2 In this case, the integrand is just the constant function 1 in any coordinate system.

Step 3 Since our description of the region in Step 1 has constant bounds on θ and variable bounds on r, we must use $dA = r\,dr\,d\theta$.

We can now put all of these elements together to get (by Fubini's Theorem)

$$A = \iint_{\text{petal}} dA = \int_{-\pi/6}^{\pi/6} \int_{0}^{\cos(3\theta)} r\,dr\,d\theta.$$

To finish things off, we just need to evaluate the iterated integral step by step:

$$\begin{aligned}
A &= \int_{-\pi/6}^{\pi/6} \int_{0}^{\cos(3\theta)} r\,dr\,d\theta \\
&= \int_{-\pi/6}^{\pi/6} \frac{1}{2} r^2 \Big|_{0}^{\cos(3\theta)} d\theta \\
&= \int_{-\pi/6}^{\pi/6} \frac{1}{2} \big(\cos(3\theta)\big)^2\,d\theta \\
&= \frac{1}{2} \int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\,d\theta \\
&= \frac{1}{2} \left[\frac{\theta}{2} + \frac{\sin(6\theta)}{12} \right]_{-\pi/6}^{\pi/6} && \text{using \#59 in the text's table of integrals} \\
&= \frac{1}{2} \left[\frac{\pi}{12} + 0 + \frac{\pi}{12} + 0 \right]]] \\
&= \frac{\pi}{12}.
\end{aligned}$$

To get some sense of this result, we might compare to the area of a circle of radius 1 which we know to be $\pi(1)^2 = \pi$. From the figure below, our result of $\pi/12$ for the area of the petal seems reasonable.