A Lagrange multiplier example

Problem: You are designing a fence to enclose a rectangular region. The enclosed area must be 1200 square meters. One side of the fence will be built using material costing 50 dollars per meter while the other three sides will be built using material costing 30 dollars per foot. Find the dimensions for building the fence at lowest cost.

Solution:

- let x and y be the dimensions of the box with x representing the length of the side to be built with more expensive material
- constraint is $xy = 1200$ which we view as level curve of the constraint function $g(x, y) = xy$
- objective function is $C(x, y) = 50x + 30x + 2 \cdot 30y = 80x + 60y$
- constrained extremes will be at points for which the constraint level curve is tangent to a level curve for the objective function
- this is equivalent to having gradient of constraint function parallel to gradient of the objective function
- gradients are parallel if there is a number λ such that $\vec{\nabla}C = \lambda\vec{\nabla}g$
- compute
$$\vec{\nabla}C = 80\,\hat{\imath} + 60\,\hat{\jmath} \qquad \text{and} \qquad \vec{\nabla}g = y\,\hat{\imath} + x\,\hat{\jmath}$$
- so, need $80\,\hat{\imath} + 60\,\hat{\jmath} = \lambda\big(y\,\hat{\imath} + x\,\hat{\jmath}\big) = \lambda y\,\hat{\imath} + \lambda x\,\hat{\jmath}$
- equating components gives
$$80 = \lambda y \qquad \text{and} \qquad 60 = \lambda x$$
- these together with the constraint give us a system of equations
$$80 = \lambda y \tag{1}$$
$$60 = \lambda x \tag{2}$$
$$xy = 1200 \tag{3}$$
- solve (1) to get $y = 80/\lambda$ and solve (2) to get $x = 60/\lambda$
- substitute these into (3) to get
$$\frac{60}{\lambda}\frac{80}{\lambda} = 1200 \quad \Longrightarrow \quad \lambda^2 = \frac{4800}{1200} = 4 \quad \Longrightarrow \quad \lambda = \pm 2$$
- use $x = 60/\lambda$ and $y = 80/\lambda$ to compute
$$x = \frac{60}{\pm 2} = \pm 30 \qquad \text{and} \qquad y = \frac{80}{\pm 2} = \pm 40$$
- from the problem context, only $x = 30$ and $y = 40$ are relevant
- so, build rectangular fence 30 meters by 40 meters (with more expensive material used on one of the 30 meter sides)