Conservative vector fields

Let $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}$ be a vector field that is continuous on some domain in $\mathbb{R}^n$. Let D be an open, connected, and simply connected region in the domain of $\vec{F}$. The following conditions are equivalent (that is, any one of the conditions implies any of the other conditions):

1. $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every smooth closed curve C in D

2. $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for every pair of smooth curves in C_1 and C_2 in D that have a common start point and a common end point.

3. There is a function V such that $\vec{\nabla} V = \vec{F}$ for all point in D.

4. The following equations hold for all points in D:

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \text{and} \qquad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

A vector field satisfying any one (and hence all) of these properties is said to be *conservative* for the region D. The second property is referred to as *path-independance*. A function V satisfying the third property with respect to $\vec{F}$ is called a *potential function* for the vector field $\vec{F}$. The fourth property is called the *component test*.