More on equations of planes in space

So far, we have seen several forms for the equation of a plane:

$Ax + By + Cz + D = 0$	standard form
$z = m_x x + m_y y + b$	slopes-intercept form
$z - z_0 = m_x(x - x_0) + m_y(y - y_0)$	point-slopes form

Using vectors, we can add another form that is coordinate-free.

A plane can be specified by giving a vector $\vec{n}$ perpendicular to the plane (called a *normal vector*) and a point P_0 on the plane. We can develop a condition or test to determine whether or not a variable point P is on the plane by thinking geometrically and using the dot product. Here's the reasoning:

- P is on the plane if and only if the vector $\overrightarrow{P_0P}$ is parallel to the plane.
- The vector $\overrightarrow{P_0P}$ is parallel to the plane if and only if $\overrightarrow{P_0P}$ is perpendicular to the normal vector $\vec{n}$.
- The vectors $\overrightarrow{P_0P}$ and $\vec{n}$ are perpendicular if and only if their dot product is zero:
$$\vec{n} \cdot \overrightarrow{P_0P} = 0.$$

So, the condition $\vec{n} \cdot \overrightarrow{P_0P} = 0$ is a new form for the equation of a line. We'll refer to this as the *point-normal form*. We can see how the point-normal form relates to our familiar forms by introducing coordinates and components. Let P_0 have coordinates (x_0, y_0, z_0), the variable point P have coordinates (x, y, z), and the normal vector $\vec{n}$ have components $\langle n_x, n_y, n_z \rangle$. With these, the vector $\overrightarrow{P_0P}$ has components $\langle x - x_0, y - y_0, z - z_0 \rangle$. So, the point-normal form can be written as

$$\begin{aligned} 0 &= \vec{n} \cdot \overrightarrow{P_0P} \\ &= \langle n_x, n_y, n_z \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\ &= n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) \\ &= n_x x + n_y y + n_z z - (n_x x_0 + n_y y_0 + n_z z_0). \end{aligned}$$

The last expression is the same as $Ax + By + Cz + D$ if we identify n_z as A, n_y as B, n_z as C and $-(n_x x_0 + n_y y_0 + n_z z_0)$ as D. This is perhaps easier to see in an example.

Example: Find the standard form for the equation of the plane that contains the point $(6, 5, 2)$ and has normal vector $\langle 7, -3, 4 \rangle$.
Solution: With (x, y, z) as the coordinates of a variable point, we can write

$$\begin{aligned} 0 &= \vec{n} \cdot \overrightarrow{P_0P} \\ &= \langle 7, -3, 4 \rangle \cdot \langle x - 6, y - 5, z - 2 \rangle \\ &= 7(x - 6) - 3(y - 5) + 4(z - 2) \\ &= 7x - 3y + 4z - 42 + 15 - 8 \\ &= 7x - 3y + 4z - 35. \end{aligned}$$

So the standard form of the equation for this plane is $7x - 3y + 4z - 35 = 0$.

Exercises

1. Use the point-normal equation to determine which, if any, of the following points are on the plane that has normal vector $2\hat{\imath} - \hat{\jmath} + 6\hat{k}$ and contains the point $(3, 4, 2)$.

   (a) $(5, -4, 0)$ (b) $(1, 6, 2)$ (c) $(2, 8, 3)$

   Answer: $(5, -4, 0)$ and $(2, 8, 3)$ are on the plane; $(1, 6, 2)$ is not

2. Find the slopes-intercept form of the equation that contains the point $(4, 2, -7)$ and has normal vector $\vec{n} = 5\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$.

   Answer: $z = -\frac{5}{2}x + \frac{3}{2}y$

3. Find the slopes-intercept form of the equation for the plane that contains the point $(4, 2, -7)$ and has normal vector $\vec{n} = \langle -6, 1, 5 \rangle$.

   Answer: $z = \frac{6}{5}x - \frac{1}{5}y - \frac{57}{5}$

   Note: The original version of this problem had $\vec{n} = \langle -6, 1, 0 \rangle$. A plane with this normal vector is vertical (since the normal vector is horizontal) and so does not have a slopes-intercept form equation.

4. Find the standard form of the equation for the plane that contains the point $(6, 3, 0)$ and is parallel to a second plane given by the equation $5x + 2y - 9z = 14$.

   Answer: $5x + 2y - 9z - 36 = 0$ or $5x + 2y - 9z = 36$

5. Find the standard form of the equation for the plane that contains the point $(7, -2, 1)$ and is perpendicular to the vector from the origin to that same point.

   Answer: $7x - 2y + z - 54 = 0$ or $7x - 2y + z = 54$