Notes on integration over a curve

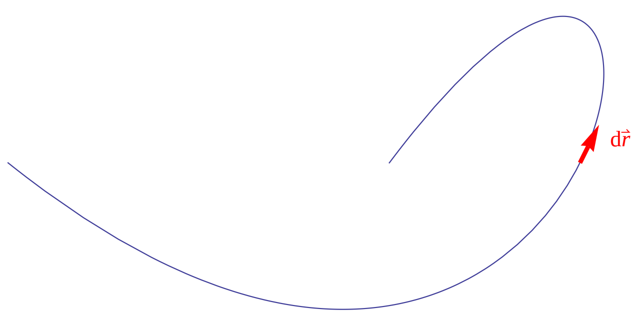
Given a curve C in the plane or in space, we can (conceptually) break it into small pieces each of which has a length ds.† In some cases, we will add up these small contributions to get the total length of the curve. We will represent this as

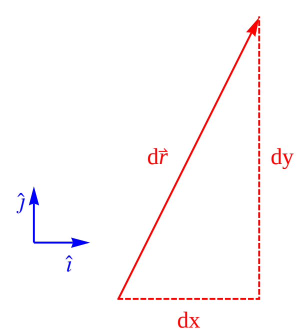
$$L = \int_C ds.$$

In other cases, we will have a length density λ defined at each point on the curve and we will add up small contributions of the form $\lambda\, ds$ to get a total (of some quantity such as charge or mass). We will represent this as

$$\text{Total} = \int_C \lambda\, ds.$$

Our general approach is to start by considering an infinitesimal displacement $d\vec{r}$ along the curve. A typical example is shown in the figure on the left below. The figure on the right below shows a closer view of $d\vec{r}$ along with components (relative to unit vectors $\hat{\imath}$ and $\hat{\jmath}$) denoted dx and dy . In terms of this coordinate system, we thus have $d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$. For a curve in space, we would express an infinitesimal displacement $d\vec{r}$ in terms of components as $d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$. In either case, the length element ds is the magnitude of the infinitesimal displacement vector. That is, $ds = \|d\vec{r}\|$.





Since we are integrating over a one-dimensional object, we will ultimately need to express ds in terms of one variable. For a given curve in the plane, dx and dy are related. To determine this relationship, we need to know how x and y are related along the curve. We can describe a curve analytically in a variety of ways such as an implicit description (i.e., an equation relating coordinates) or parametrically (i.e., formulas for the coordinates in terms of some third variable). We illustrate these for the simple case of a circle of radius R.

† We use ds rather than dL for historic reasons. Using dL for this length element would be consistent with our use of dA for area element and dV for volume element.

Example 1

A circle of radius R can be described implicitly by the equation $x^2+y^2=R^2$. So, we can "d" both sides to get a linear relation between dx and dy:

$$\mathrm{d}(x^2+y^2)=\mathrm{d}(R^2) \qquad \text{which implies} \qquad 2x\,dx+2y\,dy=0.$$

We can solve this for either dx or dy and then substitute into $d\vec{r}$. Here, we choose to solve for dy to get

$$dy=-\frac{x}{y}\,dx \qquad \text{for } y\neq 0.$$

Substituting into $d\vec{r}$, we get

$$d\vec{r}=dx\,\hat{\imath}+dy\,\hat{\jmath}=dx\,\hat{\imath}-\frac{x}{y}dy\,\hat{\jmath}=\Big(\hat{\imath}-\frac{x}{y}\,\hat{\jmath}\Big)dx \qquad \text{for } y\neq 0.$$

We can now compute

$$ds=\|d\vec{r}\|=\sqrt{1+\frac{x^2}{y^2}}\;|dx|.$$

To get the length element entirely in terms of one variable, we can solve the equation of the circle to get $y^2=R^2-x^2$ and then substitute, giving

$$ds=\|d\vec{r}\|=\sqrt{1+\frac{x^2}{R^2-x^2}}\;|dx| \qquad \text{for } x\neq \pm R.$$

With a bit of algebra, we can rewrite this as

$$ds=\|d\vec{r}\|=\frac{R}{\sqrt{R^2-x^2}}\;|dx| \qquad \text{for } x\neq \pm R.$$

Example 2

Another way to describe a circle is to think in terms of polar coordinates. We know that cartesian coordinates and polar coordinates are related by

$$x=r\cos\theta \qquad \text{and} \qquad y=r\sin\theta.$$

In polar coordinates, the equation of the circle is $r=R$ so we have

$$x=R\cos\theta \qquad \text{and} \qquad y=R\sin\theta \qquad \text{for } 0\leq\theta\leq 2\pi.$$

We can "d" each of these to get

$$dx=-R\sin\theta\,d\theta \qquad \text{and} \qquad dy=R\cos\theta\,d\theta.$$

Substituting into $d\vec{r}$, we get

$$d\vec{r}=dx\,\hat{\imath}+dy\,\hat{\jmath}=-R\sin\theta\,d\theta\,\hat{\imath}+R\cos\theta\,d\theta\,\hat{\jmath}=R(-\sin\theta\,\hat{\imath}+\cos\theta\,\hat{\jmath})d\theta.$$

We can now compute

$$ds = R\sqrt{\sin^2\theta + \cos^2\theta}|d\theta| = R\sqrt{1}|d\theta| = R|d\theta|.$$

For a circle, the relationship between ds and the $d\theta$ is much simpler than the relationship between ds and dx. This should not be a surprise since circles are natural to describe in polar coordinates. Note that the relationship we get in polar coordinates is really just the arclength formula: arclength on a circle is the product of radius and angle subtended.

In the next example, we will put our work above into use in computing a total from a length density along a curve.

Example 3

Charge is distributed on a semicircle of radius R so that the length charge density is proportional to the distance from the diameter that contains the two ends of the semicircle. Compute the total charge Q in terms of R and the maximum density λ_0.

We will compute the total charge by adding up small contributions over the semi-circle. If we let λ represent the length charge density, then the small contributions are $\lambda\, ds$ and the total charge is

$$Q = \int_{\text{semi-circle}} \lambda\, ds.$$

A picture of the specific situation for this problem is shown below.

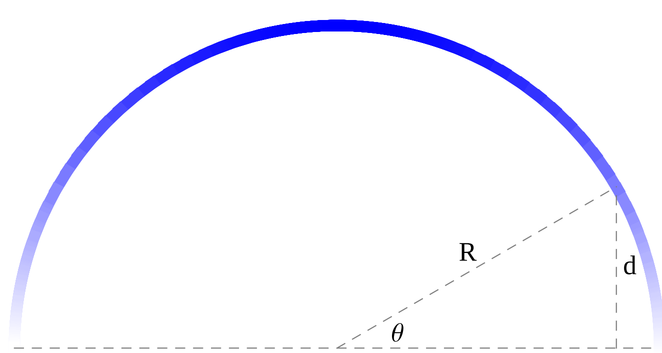
We will use polar coordinates as we did in Example 2. So, our description of the semi-circle is

$$x = R\cos\theta \qquad \text{and} \qquad y = R\sin\theta \qquad \text{for } 0 \le \theta \le \pi.$$

Using the result from Example 2, we have $ds = R|d\theta|$.

We must also determine the length charge density λ in terms of the variable θ. The length charge density is proportional to the distance d labeled in the plot. Using trigonometry, we have $d = R\sin\theta$, so

$$\lambda = kd = kR\sin\theta$$



for some proportionality constant k. The maximum density λ_0 is at the top of the semi-circle which corresponds to $\theta = \pi/2$. So, we have

$$\lambda_0 = kR\sin(\frac{\pi}{2}) = kR \qquad \text{which implies} \qquad k = \frac{\lambda_0}{R}.$$

Thus, the length density λ is related to θ by

$$\lambda = \frac{\lambda_0}{R} R\sin\theta = \lambda_0 \sin\theta.$$

So, we can express the curve integral in terms of a definite integral in the variable θ as

$$Q = \int_{\text{semi-circle}} \lambda\, ds = \int_0^{\pi} \lambda_0 \sin\theta\, R\, d\theta = R\lambda_0 \int_0^{\pi} \sin\theta\, d\theta.$$

The definite integral is easy to evaluate using the Fundamental Theorem of Calculus, giving us

$$Q = R\lambda_0 \int_0^{\pi} \sin\theta\, d\theta = R\lambda_0 \Big[-\cos\theta \Big|_0^{\pi} = R\lambda_0(1+1) = 2R\lambda_0.$$

Note that our result $Q = 2R\lambda_0$ has the correct units. We can also check that it is reasonable by comparing to some easy-to-compute quantity. Specifically, for a semi-circle with a uniform charge density of λ_0 at each point, the total charge is $\pi R\lambda_0$. Our result of $2R\lambda_0$ is less than this which is consistent with having charge density less than λ_0 at points other than the top of the semi-circle.