

Left endpoint approximation and error bound

To approximate the definite integral $\int_a^b f(x) dx$, we can use left endpoint rectangles with a total of n rectangles. We denote this approximation L_n . The error in this approximation is

$$E_n = \left| \int_a^b f(x) dx - L_n \right|.$$

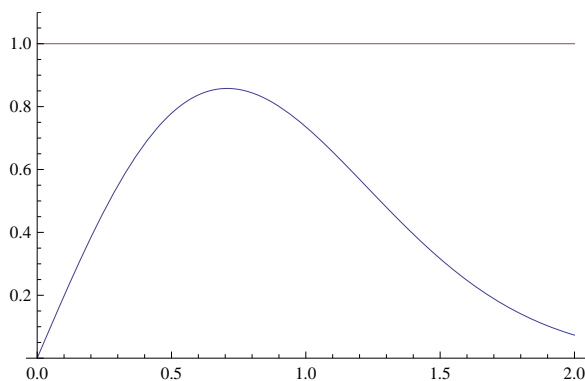
In class, we found an upper bound on this error. The upper bound B_n involves a , b , n , and some information about the derivative $f'(x)$. In particular, we need an upper bound M on $|f'(x)|$ for all x between a and b . The upper bound we found is

$$B_n = \frac{1}{2}M \frac{(b-a)^2}{n}.$$

Example: For $\int_0^2 e^{-x^2} dx$, the left endpoint approximation with four rectangles is

$$L_4 = \left(e^{-0^2} + e^{-0.5^2} + e^{-1.0^2} + e^{-1.5^2} \right) (0.5) = 1.1260$$

To get a bound on the error in this approximation, we first need a bound on $|f'(x)|$. With $f(x) = e^{-x^2}$, we compute $f'(x) = -2xe^{-x^2}$ so $|f'(x)| = 2xe^{-x^2}$ for x between 0 and 1. Below is a plot of $|f'(x)| = 2xe^{-x^2}$.



From this plot, we see that $M = 1$ is an upper bound for $|f'(x)| = 2xe^{-x^2}$. With a value for M in hand, we can compute

$$B_4 = \frac{1}{2}(1) \frac{(2-0)^2}{4} = 0.5.$$

So, the error in $L_4 = 1.1260$ is no bigger than 0.5. In other words, the exact value of the integral is somewhere between $1.1260 - 0.5 = 0.6260$ and $1.1260 + 0.5 = 1.6260$. Given that the error bound is in the tenths, we should round to tenths and report that the exact value is between 0.6 and 1.7. (Note that we have rounded the lower value down and the higher value up so that we are still guaranteed that the exact value is between the two rounded values.)

The next page has a few problems using this idea.

Problems

1. For $\int_0^2 e^{-x^2} dx$, compute an approximation using 8 left endpoint rectangles and determine a bound on the error for this approximation.

Solution: First, we compute

$$L_8 = \left[e^{-0^2} + e^{-0.25^2} + \dots + e^{-1.75^2} \right] (0.25) = 1.0044$$

In a bound on the error, we can use $M = 1$ from the example above. With this, we get

$$B_8 = \frac{1}{2}(1) \frac{(2-0)^2}{8} = 0.25.$$

So we conclude that $\int_0^2 e^{-x^2} dx = 1.004 \pm 0.25$

2. For $\int_0^2 e^{-x^2} dx$, compute an approximation using 8 *right* endpoint rectangles and determine a bound on the error for this approximation. Note: The error bound given above also applies to right endpoint approximations.

Solution: First, we compute

$$R_8 = \left[e^{-0.25^2} + e^{-0.5^2} + \dots + e^{-2.0^2} \right] (0.25) = 0.75899$$

The error bound is the same as in the previous problem, namely $B_8 = 0.25$ so we conclude that $\int_0^2 e^{-x^2} dx = 0.759 \pm 0.25$

3. For $\int_0^2 e^{-x^2} dx$, determine the number of rectangles needed to get a left endpoint approximation within a tolerance of 0.01.

Solution: Our goal is to find n such that $B_n \leq 0.01$. We know that $M = 1$ is a suitable value for this integral. With this, we have

$$B_n = \frac{1}{2}(1) \frac{(2-0)^2}{n} = \frac{2}{n}$$

so we can set up the inequality

$$\frac{2}{n} \leq 0.01$$

and solve to get

$$n \geq 200.$$

So, $n = 200$ is the smallest number of left endpoint rectangles that we can use to guarantee an approximation having error less than the tolerance of 0.01 in this case.

4. For $\int_1^3 \sin(x^2) dx$, compute an approximation using 10 left endpoint rectangles and determine a bound on the error for this approximation.

Solution: Note that $\Delta x = (3 - 1)/10 = 0.2$. The left endpoints are 1.0, 1.2, 1.4, ..., 2.8. Using these, we compute

$$L_{10} = \left[\sin(1.0^2) + \sin(1.2^2) + \sin(1.4^2) + \dots + \sin(2.8^2) \right] (0.2) = 0.48396$$

To get an error bound, we need to examine $|f'(x)|$. Here, we have $f(x) = \sin(x^2)$ so $f'(x) = 2x \sin(x^2)$. Since $|\sin \theta| \leq 1$, we have

$$|f'(x)| = |2x \sin(x^2)| = 2|x| |\sin(x^2)| \leq 2|x|(1) = 2|x|.$$

This inequality is true for all values of x . Restricting attention to the interval $[1, 3]$, we have $|x| \leq 3$ so we get

$$|f'(x)| \leq 2|x| \leq 2 \cdot 3 = 6.$$

Therefore, $M = 6$ is a suitable upper bound on $|f'(x)|$ in this case. With this, we get

$$B_{10} = \frac{1}{2}(6) \frac{(2 - 0)^2}{10} = 1.2.$$

So we conclude that $\int_1^3 \sin(x^2) dx = 0.48 \pm 1.2$

Bonus material: The error bound here is large compared with the approximation value. To get an approximation guaranteed to be closer to the exact value, we would need to use many more subintervals. Using $n = 100$ will cut the error bound down by a factor of 10 to 0.12. The L_{100} approximation turns out to be 0.467369. With this, we conclude that

$$\int_1^3 \sin(x^2) dx = 0.467 \pm 0.12.$$

Increasing n by a further factor of 10, we find $L_{1000} = 0.4637213$ and can thus conclude

$$\int_1^3 \sin(x^2) dx = 0.4637 \pm 0.012.$$