Using antidifferentiation to get a power series representation

In this example, we will get a power series representation for $\tan^{-1} x$ by antidifferentiating a known power series representation. Since

$$\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2} \tag{1}$$

we know that

$$\tan^{-1} x = \int \frac{1}{1+x^2}\, dx + C. \tag{2}$$

We also know that

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \dots \qquad \text{for } -1 < x < 1. \tag{3}$$

(Recall that we got this result by substituting $u = -x^2$ into the geometric series $\dfrac{1}{1-u} = \sum_{k=0}^{\infty} u^k$.)

Substituting this series representation for $\dfrac{1}{1+x^2}$ into (2), we get

$$\tan^{-1} x = \int \sum_{k=0}^{\infty} (-1)^k x^{2k}\, dx + C. \tag{4}$$

Theorem 20 in the text allows us to interchange the order of summation and integration in this to give

$$\tan^{-1} x = \sum_{k=0}^{\infty} \int (-1)^k x^{2k}\, dx + C. \tag{5}$$

We can move the constant factors out of the integral to get

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \int x^{2k}\, dx + C. \tag{6}$$

From the power rule, we know $\displaystyle\int x^{2k}\, dx = \frac{1}{2k+1} x^{2k+1}$. Using this in (6) gives us

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} x^{2k+1} + C. \tag{7}$$

To evaluate the constant term C, we note that $\tan^{-1} 0 = 0$. Thus $C = 0$. So, we have

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} x^{2k+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \tag{8}$$

Further analysis reveals that this equality is valid for $-1 \le x \le 1$.