

Defining definite integral

Fundamentally, a definite integral is a limit of a Riemann sum. A Riemann sum is a sum of contributions to the total. The limit in question involves the number of contributions increasing without bound while the size of each contribution goes to zero.

To make this more precise, we need more details. We start with a version that uses equal-size subintervals.

Definition (equal-size subintervals): Given a function f defined for the interval $[a, b]$, construct a *Riemann sum* in the following way:

- Partition the interval $[a, b]$ into n subintervals of equal size $\Delta x = \frac{b-a}{n}$.
- Label the subintervals with the index $k = 1, 2, 3, \dots, n$.
- Choose an input c_k in each subinterval (e.g., left endpoints, right endpoints, midpoints, ...).
- Form the Riemann sum $\sum_{k=1}^n f(c_k)\Delta x = f(c_1)\Delta x + f(c_2)\Delta x + \dots + f(c_n)\Delta x$.

If $\lim_{\Delta x \rightarrow 0} \sum_{k=1}^n f(c_k)\Delta x$ exists with the same value for all choices of inputs c_k , we say f is *integrable* for $[a, b]$ and we denote the limit $\int_a^b f(x) dx$. We call this number the *definite integral of f for $[a, b]$* .

Note: Taking $\Delta x \rightarrow 0$ is equivalent to $n \rightarrow \infty$ since $\Delta x = \frac{b-a}{n}$.

From this definition, we learn how to interpret a meaning of definite integral. Two common interpretations are *accumulation* and *area*. Here's how these might show up in applications:

- If $f(t)$ is a rate of change in some quantity for an interval of time from $t = a$ to $t = b$, then $f(t_k)\Delta t$ represents (an estimate of) a small contribution to the total accumulation. Summing and taking the limit to get $\int_a^b f(t) dt$ gives the (exact) total accumulation.
- If $f(x)$ is a function whose graph we are considering for interval from $x = a$ to $x = b$, then $f(c_k)\Delta x$ represents (an estimate of) a small contribution to the total area between the graph and the x -axis. Summing and taking the limit to get $\int_a^b f(x) dx$ gives the (exact) total area.

Note that in the area interpretation, we need to assign negative area to regions below the horizontal axis and positive area to regions above the horizontal axis.

There are contexts in which it is useful to work with more general subintervals that are not necessarily of equal size. We can generalize the previous definition.

Definition (general subintervals): Given a function f defined for the interval $[a, b]$, construct a *Riemann sum* in the following way:

- Partition the interval $[a, b]$ into n subintervals by picking a set of endpoints $P = \{x_0, x_1, x_2, \dots, x_n\}$ with $x_0 = a$ and $x_n = b$ and $x_{k-1} < x_k$.
- Label the subintervals with the index $k = 1, 2, 3, \dots, n$.
- Compute the size of each subinterval as $\Delta x_k = x_k - x_{k-1}$.
- Determine the size of the largest interval and denote this $\|P\|$. This number is called the *norm* of the partition P .
- Choose an input c_k in each subinterval (e.g., left endpoints, right endpoints, midpoints, ...).

- Form the Riemann sum $\sum_{k=1}^n f(c_k)\Delta x_k = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \dots + f(c_n)\Delta x_n$.

If $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k)\Delta x_k$ exists with the same value for all partitions P and all choices

of inputs c_k , we say f is *integrable* for $[a, b]$ and we denote the limit $\int_a^b f(x) dx$. We call this number the *definite integral of f for $[a, b]$* .

Note: What we mean by the limit as $\|P\| \rightarrow 0$ is not clear here. To make this clear, we would need a precise definition of limit. We have not developed a precise definition of limit in this course. The definition of definite integral given on p. 333 is phrased in terms of a precise definition of limit. You are not expected to understand that version of the definition.