Review of Calculus I topics

Main topics: real numbers, functions, limits, continuity, derivative, antiderivative, definite integral, Fundamental Theorems of Calculus

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Real numbers

A *rational number* is a ratio of two *integers* such as $3/25$ or $-107/5$. The rational numbers alone do not "fill up" the number line. The additional numbers needed to fill in the number line are called *irrational numbers*. The rationals and irrationals together form the set of *real numbers*.

Here is a fundamental fact about the real numbers: Between any two real numbers, there is at least one rational number and at least one irrational number.

Functions

A *function* is defined by three things: a set called the *domain*, a set called the *codomain*, and a rule that assigns one element in the codomain to each element in the domain. Each element in the domain has exactly one partner in the codomain. An element in the codomain can have no partner, one partner, or multiple partners in the domain. Those elements in the codomain that have at least one partner make up the *range* of the function.

Limits

Notation. $\lim_{x \to a} f(x) = L$

Example. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (conjecture based on numerical/graphical evidence; proof based on definition of sine function, geometry, and squeeze/pinch/sandwich theorem)

Definition (informal)**.** A function f has a *limit at a of L* if there is a single number L such that all outputs $f(x)$ are close to L for all inputs x close to a (except possibly a itself).

Definition (precise)**.** A function f has a *limit at a of L* if there is a single number L such that for any value of the challenge ϵ, there is a response δ such that $|f(x) - l| < \epsilon$ for all x satisfying $0 < |x - a| < \delta$.

Definition (informal, alternate)**.** A function f has a *limit at a of L* if for each infinite list of inputs ending at a, the corresponding list of outputs ends at L.

Computational techniques.

1. Conjecture based on numerical or graphical evidence.
2. Find algebraic expression equivalent for all $x \neq a$.
3. L'Hopital's rule (when derivatives are in place)

Applications. Used in defining continuity, derivative, definite integral, ...

Continuity

Definition. A function f is *continuous for input a* if $\lim_{x \to a} f(x) = f(a)$.

Interpretation. No holes, jumps, or poles in graph of f at a. The output $f(a)$ is what you expect based on looking at outputs for inputs near a.

Applications. Continuity is a basic, often unstated, assumption in many modeling applications.

Derivatives

Definition. A function f is *differentiable for input x* if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. If so, this limit is called the *derivative of f at x* and denoted $f'(x)$. That is,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Notation. $f'(x)$ or $\dfrac{df}{dx}$

Interpretation. For $y = f(x)$, the derivative $f'(x)$ gives the rate of change in the quantity y with respect to change in the quantity x. On a graph of the function, this is the slope of the graph. That is, slope is the rate of change in the vertical direction with respect to change in the horizontal direction.

Computational techniques. There are many computation techniques:

- Directly from definition (to get derivatives of basic functions such as power, trigonometric, exponential)

$$\begin{aligned}
\frac{d}{dx}[x^n] &= nx^{n-1} \\
\frac{d}{dx}[\sin x] &= \cos x \\
\frac{d}{dx}[\cos x] &= -\sin x \\
\frac{d}{dx}[e^x] &= e^x
\end{aligned}$$

- Rules for combinations of functions: constant factor, sum, product, quotient, chain

$$\begin{aligned}
\frac{d}{dx}[kf] &= k\frac{df}{dx} \quad \text{for any constant } k \\
\frac{d}{dx}[f+g] &= \frac{df}{dx} + \frac{dg}{dx} \\
\frac{d}{dx}[fg] &= f\frac{dg}{dx} + g\frac{df}{dx} \\
\frac{d}{dx}[f/g] &= \frac{g\,df/dx - f\,dg/dx}{g^2} \\
\frac{d}{dx}[f(g(x))] &= f'(g(x))\,g'(x)
\end{aligned}$$

- Implicit differentiation

Applications. There are many applications including

- Optimization
- Related rates
- Essential features of function graph (max/min, concavity, points of inflection)

Antiderivatives

Definition. A function F is an *antiderivative* of the function f if $F'(x) = f(x)$ for all x in the domain of f.

So the basic *antiderivative problem* is the following: Given one function f, find another function F for which the derivative is f. Write this as

$$\int f(x)\,dx = F(x)$$

Theorem. If F_1 and F_2 are antiderivatives of f for an interval if inputs, then F_1 and F_2 differ by a constant. That is, there is a constant C so that $F_1(x) - F_2(x) = C$ for all x in the relevant interval.

Computational techniques. The basic method of finding an antiderivative is to recognize patterns of derivatives. There are specific techniques to automate some of this pattern recognition. Here are some examples:

$$\frac{d}{dx}[x^2] = 2x \qquad \text{so} \qquad \int 2x\,dx = x^2 + C$$

$$\frac{d}{dx}[\sin x] = \cos x \qquad \text{so} \qquad \int \cos x\,dx = \sin x + C$$

For general powers, we have

$$\frac{d}{dx}[x^n] = nx^{n-1} \qquad \text{so} \qquad \int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$

General rules for combinations of functions include

1. $\displaystyle\int k\,f(x)\,dx = k\int f(x)\,dx$ for any constant k.
2. $\displaystyle\int \big[f(x) + g(x)\big]\,dx = \int f(x)\,dx + \int g(x)\,dx$

Definite integrals

Definition. Given a function f defined for the interval $[a, b]$, construct a *Riemann sum* in the following way: Partition the interval $[a, b]$ into n subintervals of equal size $\Delta x = \dfrac{b-a}{n}$. Label the subintervals with the index $k = 1, 2, 3, \ldots, n$. Choose an input c_k in each subinterval. Form the Riemann sum $\displaystyle\sum_{k=1}^{n} f(c_k)\Delta x$. If $\displaystyle\lim_{\Delta x\to 0}\sum_{k=1}^{n} f(c_k)\Delta x$ exists with the same value for all choices of inputs c_k, we say f is *integrable* for $[a, b]$ and we denote the limit $\displaystyle\int_a^b f(x)\,dx$. We call this number the *definite integral of f for* $[a, b]$.

Interpretation. The definite integral $\displaystyle\int_a^b f(x)\,dx$ gives the total (signed) area between the graph of the function f and the x-axis between $x = a$ and $x = b$.

If $f(t)$ is a rate of change in some quantity, then $\displaystyle\int_a^b f(t)\,dt$ gives the total accumulation of that quantity during the interval from $t = a$ to $t = b$.

If $f(x)$ is a (length) density for some quantity, then $\displaystyle\int_a^b f(x)\,dx$ gives the total amount of that quantity for the region from $x = a$ to $x = b$.

Theorem (Fundamental Theorem of Calculus 1)**.** If f is continuous for $[a, b]$ and F is defined by $F(x) = \int_a^x f(u)\,du$, then $F'(x) = f(x)$ for all x in $[a, b]$.

Theorem (Fundamental Theorem of Calculus 2)**.** If f is continuous for $[a, b]$ and F is an antiderivative of f for $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.