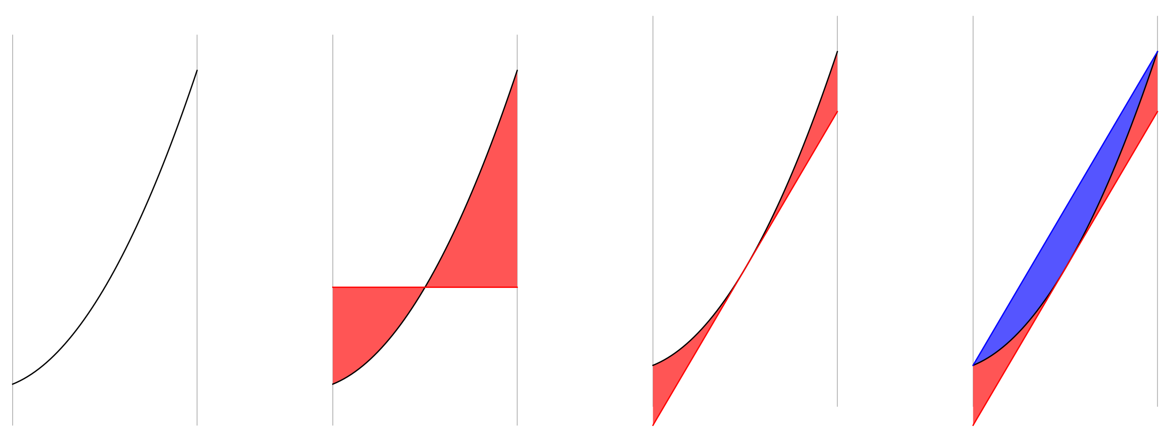
Simpson's rule as an average of midpoint and trapezoid approximations

Our general goal is to approximate a definite integral $\int_a^b f(x)\,dx$ with error less than a specified tolerance. So far, we have developed several approximation methods:

- left endpoint rectangle L_n and right endpoint rectangle R_n, each with error bounded by
$$B_n = \frac{1}{2}M_1\frac{(b-a)^2}{n} \quad \text{where } M_1 \text{ is an upper bound on } |f'(x)| \text{ for } a \le x \le b$$
- trapezoid T_n with error bounded by
$$B_n = \frac{1}{12}M_2\frac{(b-a)^3}{n^2} \quad \text{where } M_2 \text{ is an upper bound on } |f''(x)| \text{ for } a \le x \le b$$
- midpoint M_n with error bounded by
$$B_n = \frac{1}{24}M_2\frac{(b-a)^3}{n^2} \quad \text{where } M_2 \text{ is an upper bound on } |f''(x)| \text{ for } a \le x \le b$$

By looking at the errors for the trapezoid and midpoint approximations, we can deduce a still better approximation. We do this by considering a typical subinterval and the corresponding contributions to the error in trapezoid and midpoint approximations. Figure 1(a) shows the graph of a function $f(x)$ for a typical subinterval. Figure 1(b) shows the same graph along with the top of the midpoint rectangle. The region between the graph and the rectangle top is shaded. On one side of the midpoint, the rectangle is below the curve while on the other side, the rectangle is above the curve. The error is the difference between the underage and the overage. To make it easier to see the error, we can rotate the top of the midpoint rectangle top by any angle without changing the area under the resulting line. If we rotate the line so it is tangent to the curve at the midpoint, we get Figure 1(c). The tangent line lies on one side of the curve so we can clearly see the error as the area of the shaded region. In Figure 1(d), we add the trapezoid line and shade in the region that corresponds to the error in the trapezoid approximation.



(a) (b) (c) (d)

Figure 1

Comparing the error for the midpoint approximation with the error for the trapezoid approximation, we make two observations:

- the magnitude of the trapezoid error is about twice the magnitude of the midpoint error
- one approximation overshoots while the other undershoots

Based on these observations, we can deduce that a mixture of two parts midpoint approximation and one part trapezoid approximation will result in some cancellation of the individual errors to produce a better approximation. Put another way, we can take an average of three things: a midpoint approximation, a midpoint approximation, and a trapezoid approximation. To compute the average, we add the three things and divide by three. This new approximation is called *Simpson's rule*. We can express it as

$$S_n = \frac{2M_n + T_n}{3}.$$

A bound on the error in Simpson's rule gives some measure of how much better Simpson's rule is in comparison with previous approximations. We give a bound without proof. For Simpson's rule using n subintervals to approximate $\int_a^b f(x)\,dx$, the error is no bigger than the bound

$$B_n = \frac{1}{2880} M_4 \frac{(b-a)^5}{n^4} \quad \text{where } M_4 \text{ is an upper bound on } |f''''(x)| \text{ for } a \leq x \leq b$$

Important note: The number n of subintervals here differs from how subintervals are counted in the text's version of Simpson's rule. In the labeling used here, we split the interval $[a, b]$ into n pieces and denote the endpoints with $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a$ and $x_n = b$. We then denote the midpoints with $m_1, m_2, m_3, \ldots, m_n$. In Figure 2, these labels are shown below the line for the case $n = 6$. The labels above the line correspond to the text's version. In the text's version, each of the subintervals used here is considered to be *two* subintervals. So, the text's version counts twice as many subintervals. If we let $\tilde{n}$ be the text's number of subintervals, we have $\tilde{n} = 2n$. For the case $n = 6$, we get $\tilde{n} = 2 \cdot 6 = 12$. The text's subinterval endpoints are labeled $x_0, x_1, x_2, \ldots, x_{\tilde{n}}$.

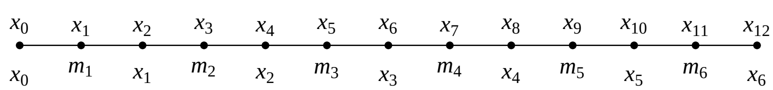


Figure 2

With the connection $\tilde{n} = 2n$, we can reconcile the error bound given here with the error bound given in Theorem 1 on page 481 of the text. Here's how to go from the text's version to the version above:

$$\frac{1}{180} M_4 \frac{(b-a)^5}{\tilde{n}^4} = \frac{1}{180} M_4 \frac{(b-a)^5}{(2n)^4} = \frac{1}{180} M_4 \frac{(b-a)^5}{16n^4} = \frac{1}{180 \cdot 16} M_4 \frac{(b-a)^5}{n^4} = \frac{1}{2880} M_4 \frac{(b-a)^5}{n^4}$$

Our approach to Simpson's rule is based on a weighted average of midpoint and trapezoid approximations. The text's approach is based on approximating the function on each subinterval with a quadratic function. This approach is a natural next step in a progression:

- Left and right endpoint approximations cap off each subinterval with a constant function (so the approximating graph consists of horizontal line segments)
- Trapezoid approximation caps off each subinterval with a *linear* function (so the approximating graph consists of line segments)
- Simpson's rule approximation caps off each subinterval with a *quadratic* function (so the approximating graph consists of parabola segments)

Both approaches lead to the same result. To see this more clearly, let's write out our version of Simpson's rule on more detail. First, note that the left and right endpoint approximations are

$$L_n = \big[f(x_0) + f(x_1) + f(x_2) + \ldots + f(x_{n-1})\big]\Delta x$$

and

$$R_n = \big[f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_n)\big]\Delta x.$$

The trapezoid approximation is

$$T_n = \frac{L_n + R_n}{2} = \frac{1}{2}\big[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\big]\Delta x.$$

With the notation introduced earlier, the midpoint approximation is

$$M_n = \big[f(m_1) + f(m_2) + f(m_3) + \ldots + f(m_n)\big]\Delta x.$$

So, Simpson's rule is

$$\begin{aligned} S_n &= \frac{2M_n + T_n}{3} \\ &= \frac{2\big[f(m_1) + f(m_2) + f(m_3) + \ldots + f(m_n)\big] + \frac{1}{2}\big[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\big]}{3}\Delta x. \end{aligned}$$

We can multiply numerator and denominator by 2 to clean this up a bit giving

$$\begin{aligned} S_n &= \frac{4\big[f(m_1) + f(m_2) + f(m_3) + \ldots + f(m_n)\big] + \big[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\big]}{6}\Delta x \\ &= \frac{f(x_0) + 4f(m_1) + 2f(x_1) + 4f(m_2) + 2f(x_2) + 4f(m_3) + \ldots + f(x_n)}{6}\Delta x. \end{aligned}$$

Compare this with the version of Simpson's rule given on page 481 of the text. Note that our midpoints m_k correspond to the text's x_l for *odd* values of the index l. (Recall Figure 2.) The only other difference is the 6 in the denominator here compared with the 3 in the text's denominator. This difference is the factor of 2 difference between the subinterval length here and the text's subinterval length.