An example of Euler's method

Given a differential equation that involves a function y and its derivative dy/dt, we can estimate the accumulation of y in small steps. To do this, we need an initial condition and we need to choose a step size.

Example 1 We will look at the differential equation

$$\frac{dy}{dt} = 0.2y$$

with the initial condition $y = 100$ for $t = 0$. We can rewrite the differential equation as

$$dy = 0.2\,y\,dt.$$

This tells us that a small change dy relates to a corresponding small change dt. If we look at a finite change Δt rather than an infinitesimal change dt, we get an approximation rather than an exact relation. In this case,

$$\Delta y \approx 0.2\,y\,\Delta t.$$

Now, starting with $y = 100$ for $t = 0$, we can compute the (approximate) change in y at $t = 0$ as

$$\Delta y \approx 0.2(100)\Delta t.$$

To get a specific value, we need to choose a step size Δt. Using $\Delta t = 0.1$, we get

$$\Delta y \approx 0.2(100)(0.1) = 2.$$

We add this to the initial value for y to get $y \approx 100 + 2 = 102$ for $t = 0 + 0.1 = 0.1$.

Using the new value of $y = 102$ for $t = 0.1$, we can compute a new (approximate) change in y. We get

$$\Delta y \approx 0.2\,y\,\Delta t = 0.2(102)(0.1) = 2.04.$$

With this, we get $y \approx 102 + 2.04 = 104.4$ for $t = 0.1 + 0.1 = 0.2$.

We can continue this process indefinitely to get an approximate value for y at any value of t. To get an approximate value of y for $t = 5$, we would need to compute a total of 50 steps using $\Delta t = 0.1$. We could choose a larger value for Δt and therefore need fewer steps to get to $t = 5$ but this would come with a loss in accuracy.

If we are going to compute a large number of steps, we might more conveniently organize our results in a table. The table below shows the calculations done above plus one additional step. The first column in the table tracks the step number n.

n	t	y	$\Delta y = 0.2\,y\,\Delta t$
0	0	100	$0.2(100)(0.1) = 2$
1	$0 + 0.1 = 0.1$	$100 + 2 = 102$	$0.2(102)(0.1) = 2.04$
2	$0.1 + 0.1 = 0.2$	$102 + 2.04 = 104.04$	$0.2(104.4)(0.1) = 2.0808$
3	$0.2 + 0.1 = 0.3$	$104.04 + 2.0808 = 106.1208$	$0.2(106.1208)(0.1) = 2.122416$

Doing these calculations by hand quickly becomes tedidous. This type of iterative calculation is easily implemented in a computer program or on a spreadsheet. The next page shows a spreadsheet calculation for 50 steps of this example along with a plot of the results. A spreadsheet like this can be easily set up using spreadsheet formulas.

Euler's method for $y' = ky$ with $y(0) = y_0$

k=	0.2
y0=	100
dt=	0.1

Step	t	y	dy
0	0.0	100.00	2.0000
1	0.1	102.00	2.0400
2	0.2	104.04	2.0808
3	0.3	106.12	2.1224
4	0.4	108.24	2.1649
5	0.5	110.41	2.2082
6	0.6	112.62	2.2523
7	0.7	114.87	2.2974
8	0.8	117.17	2.3433
9	0.9	119.51	2.3902
10	1.0	121.90	2.4380
11	1.1	124.34	2.4867
12	1.2	126.82	2.5365
13	1.3	129.36	2.5872
14	1.4	131.95	2.6390
15	1.5	134.59	2.6917
16	1.6	137.28	2.7456
17	1.7	140.02	2.8005
18	1.8	142.82	2.8565
19	1.9	145.68	2.9136
20	2.0	148.59	2.9719
21	2.1	151.57	3.0313
22	2.2	154.60	3.0920
23	2.3	157.69	3.1538
24	2.4	160.84	3.2169
25	2.5	164.06	3.2812
26	2.6	167.34	3.3468
27	2.7	170.69	3.4138
28	2.8	174.10	3.4820
29	2.9	177.58	3.5517
30	3.0	181.14	3.6227
31	3.1	184.76	3.6952
32	3.2	188.45	3.7691
33	3.3	192.22	3.8445
34	3.4	196.07	3.9214
35	3.5	199.99	3.9998
36	3.6	203.99	4.0798
37	3.7	208.07	4.1614
38	3.8	212.23	4.2446
39	3.9	216.47	4.3295
40	4.0	220.80	4.4161
41	4.1	225.22	4.5044
42	4.2	229.72	4.5945
43	4.3	234.32	4.6864
44	4.4	239.01	4.7801
45	4.5	243.79	4.8757
46	4.6	248.66	4.9732
47	4.7	253.63	5.0727
48	4.8	258.71	5.1741
49	4.9	263.88	5.2776
50	5.0	269.16	5.3832