

### Relating the exponential function to sine and cosine

For this exercise, you will probably find it easiest to work with the first five or six terms of each power series rather than working with the full power series in summation notation. For example, you will probably find it easier to work with  $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 \dots$  rather than  $e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$ .

1. (Preliminary) Recall that  $i^2 = -1$ . Simplify each of the following powers of  $i$ .

(a)  $i^3 =$                       (b)  $i^4 =$                       (c)  $i^5 =$                       (d)  $i^6 =$

2. (a) Write down the power series representation of  $e^x$  out to at least the 7th degree term.

$$e^x =$$

- (b) Do the substitution  $x = i\theta$  to get the power series representation for  $e^{i\theta}$ .

$$e^{i\theta} =$$

- (c) Use the results from 1 (and beyond) to simplify the power series in (b).

$$e^{i\theta} =$$

- (d) Organize your power series representation for  $e^{i\theta}$  from (c) by grouping together all of the terms that do not have a factor of  $i$  and grouping together all of the terms that do have a factor of  $i$ . Note that each group has infinitely many terms so you'll end each group with "+ ...".

$$e^{i\theta} =$$

- (e) Rewrite your result from (d) by factoring  $i$  from the second group of terms.

$$e^{i\theta} =$$

3. Write down the power series representation of  $\cos \theta$ .

$$\cos \theta =$$

4. Write down the power series representation of  $\sin \theta$ .

$$\sin \theta =$$

5. Copy your result from 2(e).

$$e^{i\theta} =$$

6. Compare your final result in 5 with the power series representations in 3 and 4 to reach a conclusion about the relationship among  $e^{i\theta}$ ,  $\cos \theta$ , and  $\sin \theta$ . Express this relationship in a formula.

7. Evaluate your formula from 6 for  $\theta = \pi$ .