Challenge Problems 1: Integrability

Roughly speaking, a function f is *integrable* for $[a, b]$ if the limits of all possible Riemann sums exist and are equal. For equal-sized subintervals, different choices of the inputs c_k give different Riemann sums. (For the more general case of nonequal-sized subintervals, different choices of partitions or different choices of inputs give different Riemann sums.)

For each of the following, develop a conjecture on whether the given function is integrable or not integrable for the given interval. Provide an argument or evidence to support your conjecture. Base your argument on the definition of integrable in terms of limits of Riemann sums.

1. $f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1/2 \\ 2 & \text{for } 1/2 \le x \le 1 \end{cases}$ for $[0, 1]$

2. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ for $[0, 1]$

3. $f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x \text{ is rational and } x = \dfrac{p}{q} \text{ in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ for $[1, 2]$.

4. $f(x) = \begin{cases} 0 & \text{for } x = 0 \\ \dfrac{1}{x} & \text{for } 0 < x \le 1 \end{cases}$ for $[0, 1]$

Notes:

- Proving integrability or non-integrability can be difficult in some cases. You might not have the tools to write a careful proof for each of these. For each, try to get enough understanding of the given function to make a concrete conjecture. Drawing a graph, or attempting to draw a graph, might be useful in getting some insight on the function.
- The function in Problem 1 satisfies the hypotheses of the "integrability theorem" we stated in class. For this, forget the theorem and think directly about limits of Riemann sums.
- For a few of the problems, the following fact about real numbers might be useful: *Between any two real numbers, there is at least one rational number and at least one irrational number.*
- The function in Problem 2 is discussed as Example 1 on page 335 in Section 5.3. You might want to skip over that example in text for now so you can think about your own approach.