8.6 #26

Determine whether the series $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln(n)}$ converges absolutely, converges conditionally, or diverges.

First, look at absolute convergence by analyzing the series

$$\sum_{n=2}^{\infty} \left| (-1)^{n+1} \frac{1}{n \ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

We know that the harmonic series $\sum_{n}^{\infty} \frac{1}{n}$ diverges. The additional factor of $\ln(n)$ in the denominator will only contribute very slow growth so we might conjecture that $\sum^{\infty} \frac{1}{n \ln(n)}$ also diverges. A comparison to $\sum^{\infty} \frac{1}{n^p}$ is not useful so we instead look at the related improper integral:

$$\int_2^{\infty} \frac{1}{x \ln(x)} dx = \lim_{b \to \infty} \int_2^b \frac{1}{x \ln(x)} dx = \lim_{b \to \infty} \int_{\_}^{\_} \frac{1}{u} du \quad \text{using } \begin{array}{l} u = \ln(x) \\ du = \frac{1}{x} dx \end{array}$$

$$= \lim_{b \to \infty} \ln(u) \Big|_{\_}^{\_}$$

$$= \lim_{b \to \infty} \ln(\ln x) \Big|_2^b$$

$$= \lim_{b \to \infty} \left[\ln(\ln b) - \ln(\ln 2) \right] = \infty.$$

So $\int_2^{\infty} \frac{1}{x \ln(x)} dx$ diverges and therefore so does $\sum_{n}^{\infty} \frac{1}{n \ln(n)}$.

Thus, $\sum_{n}^{\infty} (-1)^{n+1} \frac{1}{n \ln(n)}$ does <u>not</u> converges absolutely.

We can use the Alternating Series Thm to check if $\sum_{n}^{\infty} (-1)^{n+1} \frac{1}{n \ln(n)}$ converges. With $u_n = \frac{1}{n \ln(n)}$, we check the three hypotheses:

(1) $u_n > 0$ for $n \geq 2$ since $n > 0$ and $\ln(n) > 0$ for $n \geq 2$

(2) $u_{n+1} < u_n$ for $n \geq 2$ since $n < n+1$ and $\ln(n) < \ln(n+1)$

(3) $u_n \to 0$ since $n \to \infty$ and $\ln(n) \to \infty$.

Since all three hypotheses hold for $u_n = \frac{1}{n \ln(n)}$, the conclusion holds and thus $\sum^{\infty} (-1)^{n+1} \frac{1}{n \ln(n)}$ converges.

Since $\sum (-1)^{n+1} \frac{1}{n \ln(n)}$ converges but not absolutely, it converges conditionally