A precise definition of limit

What, precisely, do we mean when we declare

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{or} \qquad \lim_{x\to 5} \frac{x^2-25}{x-5} = 10 \qquad \text{or} \qquad \lim_{x\to 3} 4x = 12\ ?$$

Let's focus on the last of these for simplicity. Here, we are dealing with the function $f(x) = 4x$ for x near the fixed value $a = 3$. We declare that $\lim_{x\to 3} 4x = 12$ because we can get the outputs $f(x) = 4x$ as close to 12 as requested for all inputs x as close to 3 as needed. That is, if someone issues a challenge to get the outputs of $f(x) = 4x$ within 0.1 of 12, we can respond by showing that this happens for all inputs within 0.025 of 3. If the challenge is a smaller target around 12, we can respond with a smaller launch pad around 3 so that any input x from the launch pad generates an output $f(x)$ in the target.

The simple example of $\lim_{x\to 3} 4x = 12$ fails to illustrate one essential feature of limits: we never have to consider 3 itself as part of the launch pad. To better understand this, let's look at the statement $\lim_{x\to 5} \frac{x^2-25}{x-5} = 10$. We say this statement is true because for any challenge of a target around 10, we can find a succesful launch pad around 5. For example, if the challenge is to get within 0.1 of 10, it works to use a launch pad of 0.1 on either side of 5. But, the launch pad does not include 5 itself because 5 is not in the domain of $f(x) = \frac{x^2-25}{x-5}$.

With these examples in mind, let's look at the general statement $\lim_{x\to a} f(x) = L$. Here's a definition of what this means.

Definition (Version 1): The number L is the limit of f at a if for each target around L, there is a successful launch pad around a.

To make this completely precise, we need to specify what we mean by

- a target around L
- a launch pad around a
- a successful launch pad

A *target* around L is simply an open interval centered at L. We'll typically use ε to denote the "radius" of this interval on either side of L. So, a target with radius ε is just the open interval from $L-\varepsilon$ to $L+\varepsilon$. An output $f(x)$ is in this target if

$$L - \varepsilon < f(x) < L + \varepsilon \qquad \text{which is the same as} \qquad |f(x) - L| < \varepsilon.$$

A *launch pad* around a is simply an open interval centered at a with a taken out. We'll typically use δ to denote the "radius" of this interval on either side of a. So, a launch pad with radius δ is just the open interval from $a-\delta$ to $a+\delta$ with a taken out. An input x is in this launch pad if

$$a - \delta < x < a + \delta \quad \text{and} \quad x \neq a \qquad \text{which is the same as} \qquad 0 < |x - a| < \delta.$$

With f, a, and L specified, we can pick a target and then look at a launch pad. For a given target, a launch pad is *successful* if every input x in the launch pad has an output $f(x)$ in the target. A launch pad is *not* successful if contains any input x for which the output $f(x)$ is not in the target.

Example: Consider the function $f(x) = 4x$ for $a = 3$. For the target radius $\varepsilon = 0.1$, the launch pad with radius $\delta = 0.025$ is successful. (In fact, any launch pad with radius $\delta \leq 0.025$ is successful.) More generally, for the target radius ε, the launch pad with radius $\delta = \varepsilon/4$ is successful. Let's demonstrate this algebraically. Suppose x is in the launch pad with radius $\delta = \varepsilon/4$. Then $x \neq 3$ and

$$3 - \frac{\varepsilon}{4} < x < 3 + \frac{\varepsilon}{4}.$$

Multiplying through by 4 gives us

$$4(3 - \frac{\varepsilon}{4}) < 4x < 4(3 + \frac{\varepsilon}{4}) \qquad \text{or} \qquad 12 - \varepsilon < 4x < 12 + \varepsilon.$$

That is, $4x$ is in the target of radius ε centered at 12. So, any x in the launch pad of radius $\delta = \varepsilon/4$ centered at $a = 3$ has an output $f(x) = 4x$ in the target of radius ε centered at $L = 12$. So, for this case we can say that for any target, we have a successful launch pad. We have thus proven the limit statement $\lim_{x\to 3} 4x = 12$.

Note that the key in the previous example was to have a relationship between the target radius ε and the launch pad radius δ that guaranteed the launch pad to be successful for each possible value of ε.

Example: Prove that $\lim_{x\to 0} x^2 = 0$.
Solution: In this case, $f(x) = x^2$, $a = 0$, and $L = 0$. Let ε be a target radius. Consider the launch pad with radius $\delta = \sqrt{\varepsilon}$. Suppose x is in this launch pad of radius $\delta = \sqrt{\varepsilon}$ centered at $a = 0$. Then $x \neq 0$ and

$$0 - \sqrt{\varepsilon} < x < 0 + \varepsilon \qquad \text{or} \qquad |x - 0| < \sqrt{\varepsilon}$$

Square both sides to get $|x|^2 < \varepsilon$ or $|x^2| < \varepsilon$. Since $x^2 = x^2 - 0$, we can write the last inequality as $|x^2 - 0| < \varepsilon$. So, $f(x) = x^2$ is in the target of radius ε centered at $L = 0$. We have shown that any x in the launch pad of radius $\delta = \varepsilon$ centered at $a = 0$ has an output $f(x) = x^2$ in the target of radius ε centered at $L = 0$. We have thus proven the limit statement $\lim_{x\to 0} x^2 = 0$.

Version 1 of our definition uses some language that is not common. To connect with a more common statement of the definition, we need only unpack what we mean by target, launch pad, and successful launch pad. Here's a new version with commentary in square brackets linking to the old version.

Definition (Version 2): The number L is the limit of f at a if for each $\varepsilon > 0$ [that is, for each possible target radius], there is a corresponding number $\delta > 0$ [that is, a launch pad radius] such that

$$0 < |x - a| < \delta \qquad \text{implies} \qquad |f(x) - L| < \varepsilon$$

[that is, each x in the launch pad has $f(x)$ in the target so the launch pad is successful].

Here's a final version with the commentary removed.

Definition (Version 3): The number L is the limit of f at a if for each $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that

$$0 < |x - a| < \delta \qquad \text{implies} \qquad |f(x) - L| < \varepsilon.$$