Defining definite integral

Equal size subintervals: Given a function f defined for the interval $[a, b]$, construct a *Riemann sum* in the following way:

- Partition the interval $[a, b]$ into n subintervals of equal size $\Delta x = \dfrac{b-a}{n}$.
- Label the subintervals with the index $k = 1, 2, 3, \ldots, n$.
- Choose an input c_k in each subinterval (e.g., left endpoints, right endpoints, midpoints, ...).
- Form the Riemann sum $\displaystyle\sum_{k=1}^{n} f(c_k)\Delta x = f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$.

If $\displaystyle\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(c_k)\Delta x$ exists with the same value for all choices of inputs c_k, we say f is *integrable* for $[a, b]$ and we denote the limit $\displaystyle\int_a^b f(x)\,dx$. We call this number the *definite integral of f for $[a, b]$*.

Note: Taking $\Delta x \to 0$ is equivalent to $n \to \infty$ since $\Delta x = \dfrac{b-a}{n}$.

General subintervals (basic idea): Given a function f defined for the interval $[a, b]$, construct a *Riemann sum* in the following way:

- Partition the interval $[a, b]$ into n subintervals by picking a set of endpoints $P = \{x_0, x_1, x_2, \ldots, x_n\}$ with $x_0 = a$ and $x_n = b$ and $x_{k-1} < x_k$.
- Label the subintervals with the index $k = 1, 2, 3, \ldots, n$.
- Compute the size of each subinterval as $\Delta x_k = x_k - x_{k-1}$.
- Determine the size of the largest interval and denote this $\|P\|$. This number is called the *norm* of the partition P.
- Choose an input c_k in each subinterval (e.g., left endpoints, right endpoints, midpoints, ...).
- Form the Riemann sum $\displaystyle\sum_{k=1}^{n} f(c_k)\Delta x_k = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n$.

If $\displaystyle\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$ exists with the same value for all partitions P and all choices of inputs c_k, we say f is *integrable* for $[a, b]$ and we denote the limit $\displaystyle\int_a^b f(x)\,dx$. We call this number the *definite integral of f for $[a, b]$*.

Note: What we mean by the limit as $\|P\| \to 0$ is not clear here.

Precise definition of limit as $\|P\| \to 0$: We want to formulate a precise definition of the statement

$$\lim_{\|P\|\to 0} \sum_k^n f(c_k)\Delta x_k = I.$$

For this, we go back to the ideas of *target*, *launch pad*, and *successful launch pad.*

A *target* around I is simply an open interval centered at I. We'll typically use ϵ to denote the "radius" of this interval on either side of I. So, a target with radius ϵ is just the open interval from $I-\epsilon$ to $I+\epsilon$. A Riemann sum $\sum_k^n f(c_k)\Delta x_k$ is in this target if

$$I-\epsilon < \sum_k^n f(c_k)\Delta x_k < I+\epsilon \qquad \text{which is the same as} \qquad |\sum_k^n f(c_k)\Delta x_k - I| < \epsilon.$$

A *launch pad* in this context is simply an open interval $(0,\delta)$. The norm $\|P\|$ of a partition is in this launch pad if $\|P\| < \delta$.

With f, $[a,b]$, and I specified, we can pick a target and then look at a launch pad. For a given target, a launch pad is *successful* if every partition P with $\|P\|$ in the launch pad has Riemann sum $\sum_k^n f(c_k)\Delta x_k$ in the target for all choices of inputs c_k. A launch pad is *not* successful if there is any partition P with norm $\|P\|$ in that launch pad for which the Riemann sum $\sum_k^n f(c_k)\Delta x_k$ is not in the target for some choice of inputs c_k.

Definition (Version 1): The number I is the limit of $\sum_k^n f(c_k)\Delta x_k$ as $\|P\| \to 0$ if for each target around I, there is a successful launch pad.

Definition (Version 2): The number I is the limit of $\sum_k^n f(c_k)\Delta x_k$ as $\|P\| \to 0$ if for each $\epsilon > 0$ [that is, for each possible target radius], there is a corresponding number $\delta > 0$ [that is, a launch pad radius] such that

$$\|P\| < \delta \qquad \text{implies} \qquad |\sum_k^n f(c_k)\Delta x_k - I| < \epsilon$$

for all choices of inputs c_k [that is, each partition P with norm $\|P\|$ in the launch pad has $\sum_k^n f(c_k)\Delta x_k$ in the target for all choices of inputs c_k so the launch pad is successful].

Here's a final version with the commentary removed.

Definition (Version 3): The number I is the limit of $\sum_k^n f(c_k)\Delta x_k$ as $\|P\| \to 0$ if for each $\epsilon > 0$, there is a corresponding number $\delta > 0$ such that

$$\|P\| < \delta \qquad \text{implies} \qquad |\sum_k^n f(c_k)\Delta x_k - I| < \epsilon$$

for all choices of inputs c_k.