Name ____________________

MATH 301 Differential Equations Spring 2006 Final Exam

**Instructions:** You can work on the problems in any order. Please use just one side of each page and clearly number the problems. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem.

You can use integration aids such as a table of integrals.

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1. Find the specific solution of the initial-value problem

$$\frac{dy}{dt} = 2ty^2, \qquad y(0) = 1.$$

(16 points)

2. Find the general solution of the equation $y'(t) + y(t) = te^t$. (16 points)

3. Find the general solution of the equation $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} - 12y = 3\sin(5t)$. (16 points)

4. Consider the initial-value problem

$$4e^t \frac{d^2y}{dt^2} + \sin t \, \frac{dy}{dt} + t^3\, y = 0, \qquad y(1) = 7$$

   (a) Argue that this initial-value problem has a solution defined for all $t$. (8 points)

   (b) Argue that this initial-value problem does not have a unique solution. In fact, argue that this initial-value problem has infinitely many solutions. (8 points)

5. Consider the system of equations

$$\frac{dx}{dt} = x^2 + y^2 - 1$$
$$\frac{dy}{dt} = 2xy$$

   (a) Show that $(0, -1)$ is an equilibrium point for this system. (4 points)

   (b) Find the relevant linearized system for this equilibrium point. (6 points)

   (c) Analyze the linearized system in enough detail to draw a phase portrait. Sketch a phase portrait for the linearized system. (6 points)

   (d) Can stability of this equilibrium point be determined from the linearized system? If so, is this equilibrium point stable or unstable? (2 points)

*There is more on the flip side.*

6. Salt water is flowing through two tanks, each of volume $V$ (in gallons). Fresh water flows into each tank at a rate $r$ (in gallons per minute). Tank 2 has a drain pipe with a flow rate of $2r$. Liquid is pumped from Tank 1 to Tank 2 at a rate of $2r$ and from Tank 2 to Tank 1 at a rate of $r$. Let $y_1(t)$ be the amount of salt (in kilograms) in Tank 1 at time $t$ and let $y_2(t)$ be the amount of salt (also in kilograms) in Tank 2 at time $t$. Assume that, in each tank, the salt is always evenly mixed in the water.

   (a) Give some argument to justify the following equations as a model for this physical situation. (6 points)

$$\begin{aligned}\frac{dy_1}{dt} &= -\frac{2r}{V}y_1 + \frac{r}{V}y_2 \\ \frac{dy_2}{dt} &= \frac{2r}{V}y_1 - \frac{3r}{V}y_2\end{aligned}$$

   (b) Find the general solution of the system of equations in (a). (8 points)

   (c) What happens to $y_1$ and $y_2$ as $t \to \infty$? What is the ratio $\dfrac{y_1}{y_2}$ as $t \to \infty$? (4 points)