Name ________________________________

MATH 301 Differential Equations Spring 2006 Exam #3

Instructions:

Do your own work. You may consult your class notes and the course text. Do not consult other sources. Do not discuss generalities or specifics of the exam with anyone except me.

Turn in a complete and concise write up of your work. Show enough detail so that a peer could follow your work (both computations and reasoning). If you are not confident in some result, you will receive more credit if you make a note of this and comment on where you might be going wrong or on alternate approaches you might try.

You can use technology for calculations as needed. When you do, give some indication of what you use and how you use it.

Express each result in terms of real-valued functions.

Do any four of the five problems. Circle the problem number for each problem you submit. Each problem has a maximum value of 25 points.

The exam is due Friday, April 21 at 4:00 pm.

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1. Find the general solution of the system $\frac{d}{dt}\vec{y} = A\vec{y}$ where $A = \begin{bmatrix} 9 & 30 & 2 \\ -1 & -8 & -1 \\ -11 & -20 & -4 \end{bmatrix}$.

   Describe the behavior of solutions as $t \to \infty$.

2. Consider the system $\frac{d}{dt}\vec{y} = A\vec{y}$ where $A = \begin{bmatrix} 2\alpha & \beta \\ \beta & 0 \end{bmatrix}$. Find the general solution. Note that the solution may have different forms for different values of α and β. For each form, give the relevant value or values of the parameters. For each form of solution, make a phase portrait showing solution curves in the y_1y_2-plane.

3. Find the general solution of the system $\frac{d}{dt}\vec{y} = A\vec{y} + \vec{g}(t)$ where (22 points)

$$A = \begin{bmatrix} -7 & 5 \\ -10 & 8 \end{bmatrix} \quad \text{and} \quad \vec{g}(t) = e^{-2t}\begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

4. Consider a chain of radioactive decays $X \to Y \to Z$ where X, Y, and Z are different radioactive elements with the decay rate of X less than the decay rate of Y which in turn is less than the decay rate of Z. Let $x(t)$, $y(t)$, and $z(t)$ denote the amounts of these elements. Set up and solve an initial value problem to determine $x(t)$, $y(t)$, and $z(t)$ for a given sample if the process starts with an initial amount x_0 of element X and none of elements Y and Z. Find expressions for the ratios $y(t)/x(t)$ and $z(t)/x(t)$ and the limits of these ratios as $t \to \infty$. Discuss how you could use these results to estimate x_0 if you could only make measurements on the sample a long time after $t = 0$ (where long means in comparison to the half-lifes).

5. (a) Prove the following: Let A and B be square matrices. If A is similar to B, then e^A is similar to e^B.

   (b) The *trace* of a square matrix is defined as the sum of the diagonal entries. For a matrix A, the trace is denoted $\operatorname{tr} A$. In terms of entries a_{ij} with $1 \le i \le n$ and $1 \le j \le n$, we have

$$\operatorname{tr} A = \sum_{k=1}^{n} a_{kk} = a_{11} + a_{22} + \cdots + a_{nn}.$$

   Prove the following: If A is a diagonalizable matrix, then $\det(e^A) = e^{\operatorname{tr} A}$.

   Note: This conclusion holds for any square matrix but it's much harder to prove if the matrix is not diagonalizable.