Name ____________________

MATH 301 Differential Equations Spring 2006 Exam #1

Instructions: You can work on the problems in any order. Please use just one side of each page and clearly number the problems. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem.

You can use integration aids such as a table of integrals.

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1. Consider the differential equation $\dfrac{dy}{dt} = y\sin(y)$.
   (a) Sketch a slope field with enough detail to show all interesting features for $-\pi \le y \le \pi$. (8 points)
   (b) On your slope field, sketch solution curves. Include enough to see all possible types of solution behaviors. For each type, give the relevant initial values $y(0) = y_0$ and briefly describe the main features. (8 points)

2. Consider the initial value problem $y' + ty = t, \quad y(0) = y_0$.
   (a) Solve this as a linear first-order differential equation. (10 points)
   (b) Solve this by separation of variables. (10 points)
   (c) What is the limit of the solution as $t \to \infty$? (4 points)

3. For each of the following, solve the given differential equation or initial-value problem. (12 points each)
   (a) $(y^2e^{ty} + t) + (1+ty)e^{ty}\dfrac{dy}{dt} = 0, \qquad y(0) = 5$
   (b) $\dfrac{dy}{dt} = t^{-3}y^2, \qquad y(-1) = 2$
   (c) $\dfrac{dy}{dt} = \dfrac{2y^4 + t^4}{ty^3}$ for $t > 0$ Hint: Use the substitution $v = \dfrac{y}{t}$.

4. Show that $y(t) = \left(17t^8 - t^4\right)^{1/4}$ is the unique solution for the initial-value problem (10 points)
$$y' = \frac{2y^4 + t^4}{ty^3}, \qquad y(1) = 2.$$

5. Water is draining out of a cylindrical tank of radius R and height H through a hole located at the bottom of the tank. We want to know how long it takes to drain an initially full tank. Let $h(t)$ be the height of the water at time t with h measured in meters and t in seconds. Using some basic physics, one can show that the volume flow rate (in m^3/s) is proportional to $\sqrt{h}$. The proportionality constant depends on various factors that do not change such as the size of the hole (assumed to be small compared with H). Find an expression for the time it takes to drain the initially full tank completely or give an argument that the tank only drains completely as $t \to \infty$. (14 points)