Evolution operators

Consider the linear first-order differential equation

$$y'(t) + p(t)y(t) = g(t) \qquad \text{for } a < t < b$$

with initial condition $y(t_0) = y_0$. The general solution is

$$y(t) = y_0 e^{-P(t)} + e^{-P(t)} \int_{t_0}^{t} e^{P(s)} g(s)\, ds$$

where

$$P(t) = \int_{t_0}^{t} p(s)\, ds.$$

To emphasize the fact that the choice of antiderivative depends on the value of t_0, we will change to the notation

$$P(t_0; t) = \int_{t_0}^{t} p(s)\, ds.$$

Exercise 1. Prove each of the following identities.

1. $P(t_0; t_0) = 0$
2. $P(t_1; t_2) = -P(t_2; t_1)$
3. $P(t_1; t_2) + P(t_2; t_3) = P(t_1; t_3)$
4. $e^{P(t_1;t_2)} e^{P(t_2;t_3)} = e^{P(t_1;t_3)}$

Let's focus on the homogeneous problem

$$y'(t) + p(t)y(t) = 0 \qquad \text{with } y(t_0) = y_0 \tag{1}$$

which has the solution

$$y(t) = y_0 e^{-P(t_0;t)}$$

or

$$y(t) = e^{-P(t_0;t)} y(t_0).$$

We can interpret this last equation in the following way:

$$\boxed{\text{Multiplication by } e^{-P(t_0;t)} \text{ evolves } y \text{ from } t_0 \text{ to } t.}$$

In light of this, we will call $e^{-P(t_0;t)}$ the *evolution operator* for the homogeneous problem in Display (1).

Exercise 2. Consider three values t_1, t_2, and t_3 and note that we can write

$$y(t_2) = e^{-P(t_1;t_2)}y(t_1) \qquad \text{and} \qquad y(t_3) = e^{-P(t_2;t_3)}y(t_2).$$

Do a substitution from the first of these into the second to get a relation between $y(t_3)$ and $y(t_{1)}$. Use a result from Exercise 1 to rewrite this relationship.

Now turn attention to the nonhomogeneous problem

$$y'(t) + p(t)y(t) = g(t) \qquad \text{with } y(t_0) = y_0 \tag{2}$$

With our new notation, the solution is

$$y(t) = y_0 e^{-P(t_0;t)} + e^{-P(t_0;t)} \int_{t_0}^{t} e^{P(t_0;s)} g(s)\, ds \tag{3}$$

Exercise 3. The goal here is to rewrite the second term of the solution in Display (3). Start by moving the factor $e^{-P(t_0;t)}$ inside the integral. (Why is this justified?) Then, combine the two exponential factors using results from Exercise 1. You should be able to manipulate the second term of the solution into a form that has a nice interpretation. Do it and write out an interpretation.