Notes on line integrals

Definition of line integral

We are given a vector field $\vec{F}$ and a curve C in the domain of $\vec{F}$. The general idea of line integral is

line integral of $\vec{F}$ over curve C=
the limit of a sum of terms each having the form
(component of $\vec{F}$ tangent to C)(length of piece of C).

Here's how we make the idea precise. Break the curve C into n pieces with endpoints P_1, P_2, ..., P_{n+1}. (See Figure 1 at the end.) We can refer to these as P_i with the index i ranging from 1 to $n+1$. Define $\Delta\vec{R}_i$ to be the displacement between point P_i and point P_{i+1}. (See Figure 2.) That is, $\Delta\vec{R}_i = \overrightarrow{P_iP_{i+1}}$. At each of the points, compute the vector field output $\vec{F}(P_i)$. Recall that the dot product $\vec{F}(P_i)\cdot\Delta\vec{R}_i$ can be written

$$\vec{F}(P_i)\cdot\Delta\vec{R}_i = \|\vec{F}(P_i)\|\|\Delta\vec{R}_i\|\cos\theta = \Big(\|\vec{F}(P_i)\|\cos\theta\Big)\|\Delta\vec{R}_i\|.$$

The last expression shows that this dot product gives the component of $\vec{F}$ tangent to C times the length of a piece of C. This is what we want to add up. We define the line integral of $\vec{F}$ for the curve C as the limit of such a sum:

$$\int\limits_C \vec{F}\cdot d\vec{R} = \lim_{n\to\infty}\sum_{i=1}^{n}\vec{F}(P_i)\cdot\Delta\vec{R}_i$$

You can think of $d\vec{R}$ as an "infinitesimal" version of $\Delta\vec{R}_i$. The direction of $d\vec{R}$ is tangent to the curve at each point. (See Figure 3.)

Notation

The text often uses an alternate notation for the line integral. Here's the connection: Write the vector field $\vec{F}$ in terms of components as $\vec{F} = u\,\hat{\imath} + v\,\hat{\jmath} + w\,\hat{k}$ and write the vector $d\vec{R}$ in terms of components as $d\vec{R} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$. Here, think of dx as a small displacement parallel to the x-axis, dy as a small displacement parallel to the y-axis, and dz as a small displacement parallel to the z-axis. With these component expressions, we can write out the dot product as

$$\vec{F}\cdot d\vec{R} = \big(u\,\hat{\imath} + v\,\hat{\jmath} + w\,\hat{k}\big)\cdot\big(dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}\big) = u\,dx + v\,dy + w\,dz.$$

Using this, the notation for line integral can be written

$$\int\limits_C \vec{F}\cdot d\vec{R} = \int\limits_C u\,dx + v\,dy + w\,dz.$$

The text favors the expression on the right side and I generally use the expression on the left side.

Most of the problems are given using the notation on right side. For example, Problem 7 of Section 13.2 gives the line integral

$$\int_C (-y\,dx + x\,dy).$$

From this, you can read off that the vector field is $\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath}$.

Computing line integrals

In computing line integrals, the general plan is to express everything in terms of a single variable. This is a reasonable thing to do because a curve is a one-dimensional object. The essential things are to determine the form of $d\vec{R}$ for the curve C and the outputs $\vec{F}(P)$ along the curve C, all in terms of one variable. The displacement $d\vec{R}$ is defined to have components

$$d\vec{R} = dx\,\hat{\imath} + dy\,\hat{\jmath}$$

How to proceed depends on how we describe the curve. In general, we have two choices: a relation between the coordinates or a parametric description. The two solutions to the following example show how to work with each of these.

Example: Compute the line integral of $\vec{F}(x, y) = 3\,\hat{\imath} + 2\,\hat{\jmath}$ for the curve C that is the upper half of the circle of radius 1 traversed from left to right.

Note: To get started, you should draw a picture showing the curve and a few of the vector field outputs along the curve.

Solution 1: The equation of the circle is $x^2 + y^2 = 1$. From this, we compute

$$2x\,dx + 2y\,dy = 0.$$

Solving for dy and substituting from $x^2 + y^2 = 1$ gives

$$dy = -\frac{x}{y}\,dx = -\frac{x}{\sqrt{1-x^2}}\,dx.$$

This is the relation between dx and dy for a displacement $d\vec{R}$ along the circle. Substituting this gives

$$d\vec{R} = dx\,\hat{\imath} + dy\,\hat{\jmath} = dx\,\hat{\imath} - \frac{x}{\sqrt{1-x^2}}\,dx\,\hat{\jmath} = \left(\hat{\imath} - \frac{x}{\sqrt{1-x^2}}\,\hat{\jmath}\right) dx$$

The vector field here is constant so all outputs along the curve C are $\vec{F}(P) = 3\hat{\imath} + 2\hat{\jmath}$. We thus have

$$\vec{F} \cdot d\vec{R} = \left(3\,\hat{\imath} + 2\,\hat{\jmath}\right) \cdot \left(\hat{\imath} - \frac{x}{\sqrt{1-x^2}}\,\hat{\jmath}\right) dx = \left(3 - \frac{2x}{\sqrt{1-x^2}}\right) dx$$

This is the integrand. For the curve C, the variable x ranges from -1 to 1, so we have

$$\int_C \vec{F} \cdot d\vec{R} = \int_{-1}^{1} \left(3 - \frac{2x}{\sqrt{1-x^2}} \right) dx = \text{ some work to be done here } = 6.$$

Solution 2: We parametrize the curve by

$$x = -\cos t \qquad \text{and} \qquad y = \sin t \qquad \text{for} \qquad 0 \leq t \leq \pi.$$

You should confirm that this traces out the curve C in the correct direction (from left to right). From these, we compute

$$dx = \sin t \, dt \qquad \text{and} \qquad dy = \cos t \, dt.$$

Substituting into $d\vec{R}$ gives

$$d\vec{R} = dx \, \hat{\imath} + dy \, \hat{\jmath} = \sin t \, dt \, \hat{\imath} + \cos t \, dt \hat{\jmath} = \big(\sin t \, \hat{\imath} + \cos t \hat{\jmath}\big) \, dt.$$

The vector field here is constant so all outputs along the curve C are $\vec{F}(P) = 3\,\hat{\imath} + 2\,\hat{\jmath}$. We thus have

$$\vec{F} \cdot d\vec{R} = \big(3\,\hat{\imath} + 2\,\hat{\jmath}\big) \cdot \big(\sin t \, \hat{\imath} + \cos t \, \hat{\jmath}\big) \, dt = \big(3 \sin t + 2 \cos t\big) \, dt.$$

This is the integrand. For the curve C, the variable t ranges from 0 to π, so we have

$$\int_C \vec{F} \cdot d\vec{R} = \int_0^{\pi} \big(3 \sin t + 2 \cos t\big) \, dt = \text{ some work to be done here } = 6.$$

Comments: In comparing the two solutions, you might think that the algebra is more complicated in Solution 1. This is probably so. The advantage of Solution 1 is that we all know the equation of a circle is $x^2 + y^2 = 1$. For Solution 2, to get started, we need to parametrize the curve. This is not too bad for a circle. The choice of which style to use depends on personal preference and the easiest way to describe a given curve.

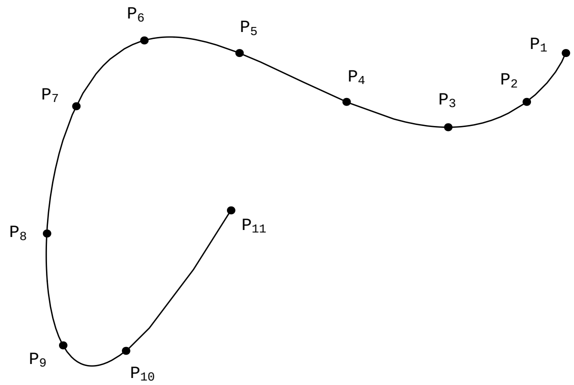

Figure 1. The curve C broken into pieces with endpoint P_i.

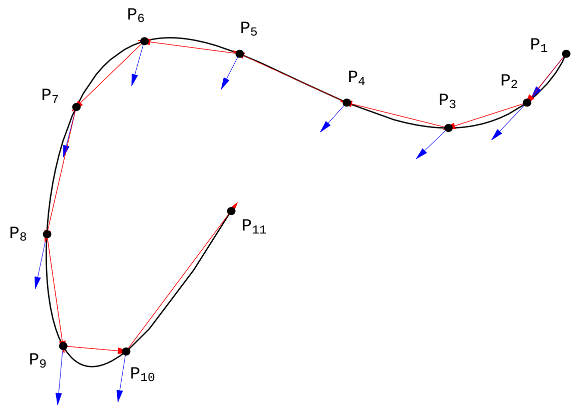

Figure 2. The curve C with the vectors $\Delta\vec{R}_i$ (in red) and $\vec{F}(P_i)$ (in blue).

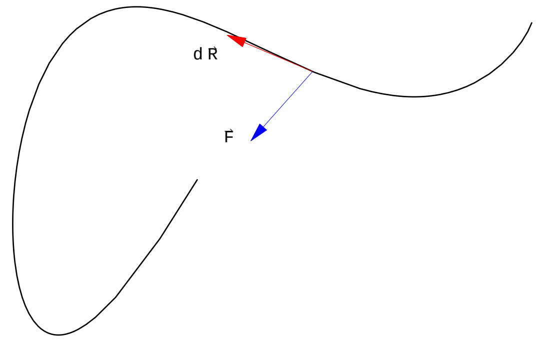

Figure 3. The curve C with an example of $d\vec{R}$ and $\vec{F}$ at a point.