A hint on Section 1.2, Problem 19

We are looking at a solution $y(t)$ in the form

$$y(t) = \frac{1}{\mu(t)}\left(\mu(t_0)y_0 + \int_{t_0}^{t} \mu(s)f(s)\,ds\right) = \frac{\mu(t_0)y_0}{\mu(t)} + \frac{\displaystyle\int_{t_0}^{t} \mu(s)f(s)\,ds}{\mu(t)}$$

In class, we had enough to show that the first term in this last expression has a limit of 0 as $t \to \infty$. To deal with the second term, consider two cases:

Case 1: $\displaystyle\int_{t_0}^{t} \mu(s)f(s)\,ds$ is bounded for all t. This is equivalent to saying there is a constant M such that

$$\int_{t_0}^{t} \mu(s)f(s)\,ds \leq M \quad \text{for all } t.$$

Case 2: $\displaystyle\int_{t_0}^{t} \mu(s)f(s)\,ds$ is not bounded for all t. This is equivalent to saying

$$\lim_{t\to\infty} \int_{t_0}^{t} \mu(s)f(s)\,ds = \infty.$$

Case 1 is straightforward to deal with. For Case 2, we can apply L'Hôpital's rule since we are dealing with a limit that has the indeterminate form $\frac{\infty}{\infty}$. You should finish both cases. Keep in mind that we have

$$\mu(t) = \exp\left(\int_{t_0}^{t} a(x)\,dx\right).$$