Population models

1. *Consider a population that has a constant per capita rate of change in its natural state. Suppose humans harvest this population at a constant rate. (As an example, you might think of fishing regulations that allow a fixed number of fish to be caught each year.)*

Let $p(t)$ give the population at time t. Let r be the constant per capita rate of change so in the natural state, we have

$$\frac{p'(t)}{p(t)} = r \qquad \text{or} \qquad p'(t) = rp(t).$$

Let h be the constant harvest rate. This is a negative contribution to the overall rate of change $p'(t)$ so with harvesting included, we have

$$p'(t) = rp(t) - h \qquad \text{or} \qquad \frac{dp}{dt} = rp - h.$$

To solve this, first do some algebra to separate variables giving

$$\frac{1}{rp-h}\,dp = dt$$

so

$$\int \frac{1}{rp-h}\,dp = \int dt.$$

The right side is easy to evaluate. For the left side, factor r from the denominator giving

$$\int \frac{1}{rp-h}\,dp = \frac{1}{r}\int \frac{1}{p-\frac{h}{r}}\,dp = \frac{1}{r}\ln\left|p-\frac{h}{r}\right|.$$

We thus have

$$\frac{1}{r}\ln\left|p-\frac{h}{r}\right| = t + C.$$

Now we want to solve for p. First, multiply both sides by r to get

$$\ln\left|p-\frac{h}{r}\right| = rt + C$$

where we rename rC as C. This is equivalent to the exponential form

$$\left|p-\frac{h}{r}\right| = e^{rt+C} = e^{rt}e^{C} = Ce^{rt}$$

where we rename e^C as C in the last step. We can drop the absolute value by absorbing $\pm$ into the constant C giving

$$p - \frac{h}{r} = Ce^{rt}$$

Thus

$$p(t) = Ce^{rt} + \frac{h}{r}.$$

We can now solve for the constant C in terms of the initial population p_0 by evaluating this for $t = 0$. This gives

$$p(0) = Ce^0 + \frac{h}{r} = C + \frac{h}{r}$$

to be compared with

$$p(0) = p_0.$$

We must thus have

$$C + \frac{h}{r} = p_0 \qquad \text{or} \qquad C = p_0 - \frac{h}{r}.$$

Substituting this into the general solution gives us

$$p(t) = \left(p_0 - \frac{h}{r}\right)e^{rt} + \frac{h}{r}.$$

We can see that this is consistent with what we saw using the slope field. If $p_0 = \frac{h}{r}$, the first term is zero and $p(t) = \frac{h}{r}$ for all t. If $p_0 > \frac{h}{r}$, the first term is positive and gives exponential growth. If $p_0 < \frac{h}{r}$, the first term is negative and results in a population of zero in finite time. This time T is found by solving

$$0 = \left(p_0 - \frac{h}{r}\right)e^{rT} + \frac{h}{r}.$$

The result is

$$T = \frac{1}{r}\ln\left(\frac{h}{h - rp_0}\right).$$

2. *Consider a population model in which the per capita rate of change is proportional to the difference between a constant K and the population. (The constant K is usually called the* carrying capacity *of the environment.)*

We'll just set up the differential equation here. Solving it involves some messy algebra. We'll do this in class on Monday. Don't worry about it for the exam.

Let $p(t)$ be the population at time t. The difference between K and the population is given by $K - p(t)$. If the per capita rate of change is proportional to this difference, we can write

$$\frac{p'(t)}{p(t)} = a\big(K - p(t)\big)$$

where a is the proportionality constant. Thus,

$$p'(t) = ap(t)\big(K - p(t)\big) \qquad \text{or} \qquad \frac{dp}{dt} = ap(K - p).$$

To draw a slope field, note that the right side is zero for both $p = 0$ and $p = K$. Thus, we will have horizontal tangent line segments at both $p = 0$ and $p = K$.