Line integrals: Definition and evaluation

Suppose an object is displaced from point P to point Q along a curve C. A force $\vec{F}$ is exerted on the object during the displacement. (This force need not be the net force.) We want to calculate the work done by this force for the displacement.

We know that for the special case of a straight displacement from the point P to the point Q and a constant force $\vec{F}_0$, the work is given by $\vec{F}_0 \cdot \overrightarrow{PQ}$. (Note: $\overrightarrow{PQ}$ is the vector that starts at point P and ends at point Q.)

To use our knowledge about work for the straight displacement/constant force case, we split the curve into pieces, each of which is small enough to be approximately straight. Specifically, break the curve C into n pieces with endpoints P_1, P_2, $\ldots$, P_{n+1}. (See Figure 1.) We can refer to these as P_i with the index i ranging from 1 to $n+1$.

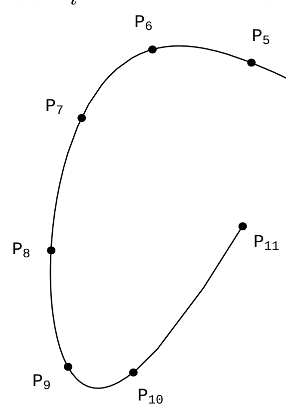

Figure 1. The curve C broken into pieces with endpoints P_i.

Define $\Delta\vec{R}_i$ to be the displacement between point P_i and point P_{i+1}. (See Figure 2.) That is, $\Delta\vec{R}_i = \overrightarrow{P_iP_{i+1}}$. At each of the points, we have the force $\vec{F}(P_i)$.

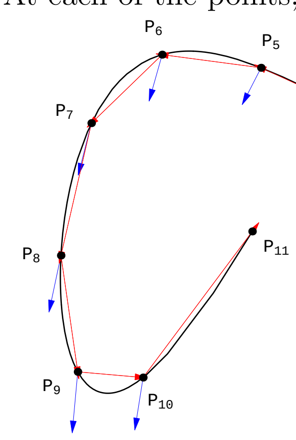

Figure 2. The curve C with the vectors $\Delta\vec{R}_i$ (in red) and $\vec{F}(P_i)$ (in blue).

The work done for the displacement $\Delta\vec{R}_i$ is approximately

$$\Delta W_i \approx \vec{F}(P_i)\cdot\Delta\vec{R}_i.$$

The total work for the entire displacement along the curve C is approximated by the sum of these ΔW_i so

$$W \approx \sum_{i=1}^{n} \vec{F}(P_i)\cdot\Delta\vec{R}_i.$$

The exact work is given by the limit as $n\to\infty$ so

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(P_i)\cdot\Delta\vec{R}_i.$$

This limit is *defined* as *the line integral of* $\vec{F}$ *for the curve* C and denoted

$$\int\limits_C \vec{F}\cdot d\vec{R} \qquad \text{or} \qquad \int_P^Q \vec{F}\cdot d\vec{R}.$$

We thus have

$$W = \int\limits_C \vec{F}\cdot d\vec{R}.$$

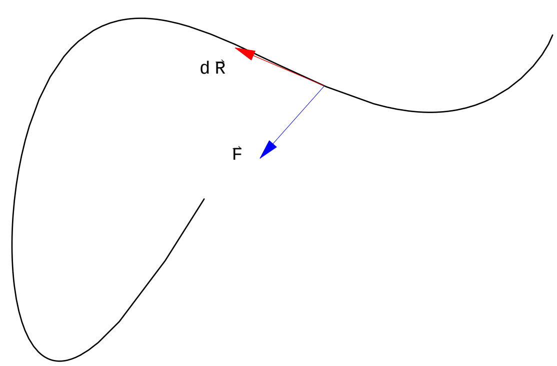

Figure 3. The curve C with an example of $d\vec{R}$ and $\vec{F}$ at a point.

There are many ways to evaluate a given line integral. Three methods are illustrated for the following example.

Example: A block on a table is pushed once around a circle of radius r_0 by a force applied horizontally. Compute the work done on the block by friction for this displacement.

Solution 1: Let $\vec{f}$ denote the friction force. The magnitude of the friction force is constant. The direction of the friction force is always opposite to the displacement. For any displacement $d\vec{R}$, we thus have

$$\vec{f}\cdot d\vec{R} = \|\vec{f}\|\|d\vec{R}\|\cos(\pi) = f\,dR\,(-1) = -f\,dR$$

where we use the usual convention of $\|\vec{f}\| = f$ and $\|d\vec{R}\| = dR$. The work is thus given by

$$W = \int_{\text{circle}} \vec{f} \cdot d\vec{R} = -\int_{\text{circle}} f\, dR = -f \int_{\text{circle}} dR$$

The integral that remains corresponds to summing up the magnitude of the displacement along the curve. This is just the length L of the curve which in this case is $L = 2\pi r_0$. Thus, we have

$$W = -f(2\pi r_0) = -2\pi r_0 f.$$

In many cases, we model the magnitude of the friction force by $f = \mu N$ where μ is a constant (depending on the object and the surface) and N is the magnitude of the normal force. If the only forces in the vertical direction are gravity and the normal force, then $N = mg$ where m is the mass of the object. In this case, $f = \mu mg$ and the work done by friction can be expressed as

$$W = -2\pi r_0 \mu mg.$$

Solution 2: We can parametrize the circle by $\vec{R}(\theta) = r_0 \cos\theta\, \hat{\imath} + r_0 \sin\theta\, \hat{\jmath}$ with θ ranging from 0 to 2π for one revolution. From this, we compute

$$\frac{d\vec{R}}{d\theta} = -r_0 \sin\theta\, \hat{\imath} + r_0 \cos\theta\, \hat{\jmath}$$

to get

$$d\vec{R} = r_0(-\sin\theta\, \hat{\imath} + \cos\theta\, \hat{\jmath})\, d\theta.$$

At the point on the circle corresponding to angle θ, the friction force is given by

$$\vec{f} = f \sin\theta\, \hat{\imath} - f \cos\theta\, \hat{\jmath} = f(\sin\theta\, \hat{\imath} - f \cos\theta\, \hat{\jmath}).$$

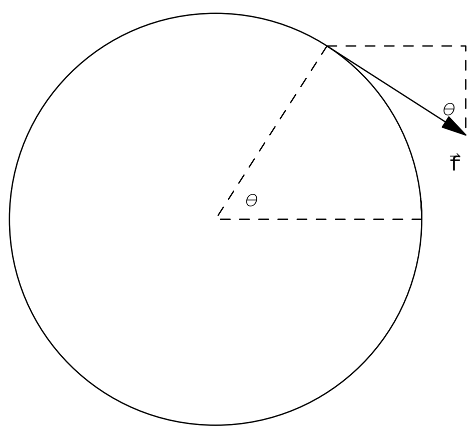

Figure 4. Geometry to determine components of the friction force vector.

With component expressions for $\vec{f}$ and $d\vec{R}$, we compute

$$\vec{f} \cdot d\vec{R} = f r_0(-\sin^2\theta - \cos^2\theta)\, d\theta = -f r_0(\sin^2\theta + \cos^2\theta)\, d\theta = -f r_0\, d\theta.$$

The work done is thus

$$W = \int\limits_{\text{circle}} \vec{f} \cdot d\vec{R} = \int_0^{2\pi} -fr_0 \, d\theta = -fr_0 \int_0^{2\pi} d\theta = -fr_0(2\pi) = -2\pi r_0 f = -2\pi r_0 \mu mg.$$

Solution 3: In cartesian coordinates, the equation of the circle is $x^2 + y^2 = r_0^2$. Differentiating both sides with respect to x gives

$$2x + 2y\frac{dy}{dx} = 0$$

so

$$2x\,dx + 2y\,dy = 0.$$

For a displacement along the circle, this relates a horizontal change dx to the corresponding vertical change dy. We can solve to get

$$dy = -\frac{x}{y}\,dx \qquad \text{for } y \neq 0.$$

With this, we can write $d\vec{R} = dx\,\hat{\imath} + dy\,\jmath$ as

$$d\vec{R} = dx\,\hat{\imath} - \frac{x}{y}\,dx\,\hat{\jmath} = \left(\hat{\imath} - \frac{x}{y}\,\hat{\jmath}\right) dx.$$

for $y \neq 0$.

To express the friction force vector in cartesian coordinates, we use $\sin\theta = y/r_0$ and $\cos\theta = x/r_0$ to write

$$\vec{f} = f(\sin\theta\,\hat{\imath} - f\cos\theta\,\hat{\jmath}) = f\left(\frac{y}{r_0}\,\hat{\imath} - \frac{x}{r_0}\,\hat{\jmath}\right) = \frac{f}{r_0}(y\,\hat{\imath} - x\,\hat{\jmath}).$$

With component expressions for $\vec{f}$ and $d\vec{R}$, we compute

$$\vec{f} \cdot d\vec{R} = \frac{f}{r_0}\left(y + \frac{x^2}{y}\right) dx = \frac{f}{r_0}\left(\frac{y^2 + x^2}{y}\right) dx.$$

We can use the relation $x^2 + y^2 = r_0^2$ to express this entirely in terms of x as

$$\vec{f} \cdot d\vec{R} = \frac{f}{r_0}\left(\frac{r_0^2}{\pm\sqrt{r_0^2 - x^2}}\right) dx = \pm\frac{fr_0}{\sqrt{r_0^2 - x^2}}\,dx.$$

Here, the positive root corresponds to the upper half of the circle and the negative root corresponds to the lower half.

With this, we can now compute the work done as

$$W = \int\limits_{\text{circle}} \vec{f}\cdot d\vec{R} = \int\limits_{\text{upper half}} \vec{f}\cdot d\vec{R} + \int\limits_{\text{lower half}} \vec{f}\cdot d\vec{R} = \int_{r_0}^{-r_0} \frac{fr_0}{\sqrt{r_0^2 - x^2}}\,dx + \int_{-r_0}^{r_0} -\frac{fr_0}{\sqrt{r_0^2 - x^2}}\,dx.$$

Switching the limits of integration in the first integral gives us

$$W = -2\int_{-r_0}^{r_0} \frac{fr_0}{\sqrt{r_0^2 - x^2}}\,dx = -2fr_0 \int_{-r_0}^{r_0} \frac{1}{\sqrt{r_0^2 - x^2}}\,dx = -2fr_0\,\sin^{-1}\left(\frac{x}{r_0}\right)\Bigg|_{-r_0}^{r_0}$$

$$= -2fr_0\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = -2\pi r_0 f = -2\pi r_0 \mu mg.$$

Problems

1. A force given by $\vec{F} = Ay\,\hat{\imath}$ with $A > 0$ is exerted on an object as it is displaced from the point $P(a, a)$ to the point $Q(-a, a)$. Compute the work done by this force for each of the following.

   (a) The object is displaced along the straight line from P to Q.

   (b) The object is displaced along the parabola $ay = x^2$ from P to Q.

   Check that your results makes sense in terms of units and sign.

2. A force given by $\vec{F} = Az\,\hat{\imath} + \dfrac{B}{\beta^2 + z^2}\,\hat{k}$ is exerted on an object as it is displaced from the point $P(a, 0, 0)$ to the point $Q(a, 0, b)$. Compute the work done by this force for each of the following. Assume the constants A, B and β are all positive.

   (a) The object is displaced along the straight line from P to Q.

   (b) The object is displaced along one full turn of a helix with constant pitch from P to Q.

   Check that your results makes sense in terms of units and sign.