Another example using a precise definition of limit (Version 2)

Definition:
Let f be a function whose domain contains a set of the form $\{x \big| 0 < |x-a| < r\}$ for some r. The number L is *the limit of f at a* if there is a positive function $\delta(\varepsilon)$ with domain $(0,\infty)$ sch that $0 < |x-a| < \delta(\varepsilon)$ implies that $|f(x) - L| < \varepsilon$.

Example: Prove the following limit property:
If $\lim_{x\to a} p(x) = L_1$ and $\lim_{x\to a} q(x) = L_2$, then

$$\lim_{x\to a}\big(p(x)+q(x)\big) = \lim_{x\to a} p(x) + \lim_{x\to a} q(x) = L_1 + L_2.$$

Solution: The conclusion we need to reach is the limit statement

$$\lim_{x\to a}\big(p(x)+q(x)\big) = L_1 + L_2.$$

In the notation of the definition, we have $f(x) = p(x) + q(x)$ and $L = L_1 + L_2$.

We assume that $\lim_{x\to a} p(x) = L_1$ and $\lim_{x\to a} q(x) = L_2$. From the definition of $\lim_{x\to a} p(x) = L_1$, there is a positive function $\delta_1(\varepsilon)$ such that

$$0 < |x-a| < \delta_1(\varepsilon) \qquad \text{implies that} \qquad |p(x) - L_1| < \varepsilon.$$

Likewise, from the definition of $\lim_{x\to a} q(x) = L_2$, there is a positive function $\delta_2(\varepsilon)$ such that

$$0 < |x-a| < \delta_2(\varepsilon) \qquad \text{implies that} \qquad |q(x) - L_2| < \varepsilon.$$

Let $\delta(\varepsilon) = \min\Big\{\delta_1(\tfrac{\varepsilon}{2}), \delta_2(\tfrac{\varepsilon}{2})\Big\}$. Note that $\delta(\varepsilon)$ is defined for each ε in the interval $(0,\infty)$ since $\frac{\varepsilon}{2} > 0$ whenever $\varepsilon > 0$. Note also that $\delta(\varepsilon) > 0$ for each such ε since $\delta_1(\frac{\varepsilon}{2}) > 0$ and $\delta_2(\frac{\varepsilon}{2}) > 0$.

Now assume $0 < |x-a| < \delta(\varepsilon)$. By the choice of $\delta(\varepsilon)$, this implies

$$0 < |x-a| < \delta_1(\tfrac{\varepsilon}{2}) \qquad \text{and} \qquad 0 < |x-a| < \delta_2(\tfrac{\varepsilon}{2}).$$

Thus,

$$|p(x) - L_1| < \tfrac{\varepsilon}{2} \qquad \text{and} \qquad |q(x) - L_2| < \tfrac{\varepsilon}{2}$$

Adding corresponding sides of these inequalitites gives us

$$|p(x) - L_1| + |q(x) - L_2| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2}$$

so

$$|p(x) - L_1| + |q(x) - L_2| < \varepsilon.$$

For any expressions A and B, it is true that $|A+B| \le |A| + |B|$. Using this with $A = p(x) - L_1$ and $B = q(x) - L_2$, we get

$$|(p(x) - L_1) + (q(x) - L_2)| < \varepsilon.$$

Rearranging the terms on the left side of this gives

$$|(p(x) + q(x)) - (L_1 + L_2)| < \varepsilon.$$

Thus, there is a function $\delta(\varepsilon)$ such that

$$0 < |x-a| < \delta(\varepsilon) \qquad \text{implies that} \qquad |f(x) - L| = \big|\big(p(x)+q(x)\big) - (L_1 + L_2)\big| < \varepsilon.$$

Problem: Prove the constant multiple property for limits.