A precise definition of limit

Definition:
Let f be a function whose domain contains a set of the form $\{x|0 < |x-a| < r\}$ for some r. The number L is *the limit of f at a* if there is a positive function $\delta(\varepsilon)$ with domain $(0, \infty)$ sch that $0 < |x-a| < \delta(\varepsilon)$ implies that $|f(x) - L| < \varepsilon$.

Notation:
If the number L is the limit of the function f at a, we denote this by $\lim_{x \to a} f(x) = L$.

Comment:
The condition that $\delta(\varepsilon)$ has domain $(0, \infty)$ says that $\delta(\varepsilon)$ is defined for any $\varepsilon > 0$. Saying that $\delta(\varepsilon)$ is a positive function means that $\delta(\varepsilon) > 0$ for any $\varepsilon > 0$. This is equivalent to saying that the range of $\delta(\varepsilon)$ is contained in the interval $(0, \infty)$.

Example: Prove that $\lim_{x \to 1}(4x+1) = 5$.
Solution: In this case, $f(x) = 4x+1$, $a = 1$, and $L = 5$.

Let $\delta(\varepsilon) = \frac{\varepsilon}{4}$. Note that $\delta(\varepsilon)$ is defined for $\varepsilon > 0$ and that $\delta(\varepsilon) > 0$ for each $\varepsilon > 0$.

Now assume $0 < |x-1| < \delta(\varepsilon)$. Thus $|x-1| < \frac{\varepsilon}{4}$. Multiply both sides by 4 to get $4|x-1| < \varepsilon$ or $|4x-4| < \varepsilon$. Since $-4 = 1-5$, we can write the last inequality as $|4x+1-5| < \varepsilon$. This is equivalent to $|f(x) - L| < \varepsilon$.

We have shown that $\delta(\varepsilon) = \frac{\varepsilon}{4}$ is a positive function with domain $(0, \infty)$ such that $0 < |x-1| < \delta(\varepsilon)$ implies $|f(x) - 5| < \varepsilon$ for the function $f(x) = 4x+1$. We have thus proven that $\lim_{x \to 1}(4x+1) = 5$.

Example: Prove that $\lim_{x \to 0} x^2 = 0$.
Solution: In this case, $f(x) = x^2$, $a = 0$, and $L = 0$.

Let $\delta(\varepsilon) = \sqrt{\varepsilon}$. Note that $\delta(\varepsilon)$ is defined for $\varepsilon > 0$ and that $\delta(\varepsilon) > 0$ for each $\varepsilon > 0$.

Now assume $0 < |x-0| < \delta(\varepsilon)$. Thus $|x| < \sqrt{\varepsilon}$. Square both sides to get $|x|^2 < \varepsilon$ or $|x^2| < \varepsilon$. Since $x^2 = x^2 - 0$, we can write the last inequality as $|x^2 - 0| < \varepsilon$. This is equivalent to $|f(x) - L| < \varepsilon$.

We have shown that $\delta(\varepsilon) = \sqrt{\varepsilon}$ is a positive function with domain $(0, \infty)$ such that $0 < |x-0| < \delta(\varepsilon)$ implies $|f(x) - 0| < \varepsilon$ for the function $f(x) = x^2$. We have thus proven that $\lim_{x \to 0} x^2 = 0$.

Problems:

1. Prove $\lim_{x \to 2}(3x-1) = 5$.
2. Prove $\lim_{x \to 4}(6-2x) = -2$.
3. Prove $\lim_{x \to 5} \frac{x}{10} = \frac{1}{2}$.
4. Prove $\lim_{x \to 0} x^3 = 0$.