Physics 121MA/Math 122PH
Fall 2002 Project 4 Name ______________________________

Instructions: We encourage you to work with others in a small group on this project. You should write your solution neatly using complete sentences that incorporate all symbolic mathematical expressions into the grammatical structure. Include enough detail to allow a fellow student to reconstruct your work, but you need not show every algebraic or arithmetic step. It is important that you do your own writing, even if you have worked out the details with other people. All graphs should be done carefully on graph paper or drawn by a computer. This project is due at the beginning of class on Tuesday, November 19.

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1. The **midpoint approximation** of a definite integral $\int_a^b f(x)\ dx$ is the rectangular approximation in which the height of the rectangle on the ith subinterval is given by the output of the function for the midpoint of the subinterval. For the subinterval $[c, d]$, the area of the approximating rectangle is $f\left(\frac{c+d}{2}\right)\Delta x$. We'll use Mid_n to denote the midpoint approximation with n subintervals.

   (a) Illustrate the midpoint approximation for a generic subinterval. Give an argument to justify the following statement: *The area of the midpoint rectangle is equal to the area of a trapezoid in which the top of the trapezoid is given by the tangent line to the graph of f at the midpoint $\left(\frac{c+d}{2}, f\left(\frac{c+d}{2}\right)\right)$.* (*Hint:* Think about rotating the top of the rectangle around the point $\left(\frac{c+d}{2}, f\left(\frac{c+d}{2}\right)\right)$.)

   (b) Derive an upper bound on the error for the midpoint approximation Mid_n in terms of a, b, n, and M_1 where is an upper bound for $|f'(x)|$.

2. We next look at comparing the trapezoid and midpoint approximations.

   (a) Use pictures to compare the error in the original trapezoid approximation with the error in the midpoint approximation for a generic subinterval. For this it is useful to use the "tangent line trapezoid" view of the midpoint approximation. Make a conjecture about the relative sign of the two errors and the relative magnitude of the two errors.

   (b) Compute the trapezoid approximation and the midpoint approximation (with the same number of subintervals) for at least three examples for which you can also compute an exact value using the Second Fundamental Theorem. (Choose at least two non-polynomial functions.) Use the exact value to compute the actual total errors. Compare these errors to the conjecture you made in (b).

   (c) Compare the error bound expression for the trapezoid approximation (as found in the text) with the error bound expression for the midpoint approximation (as found above).

3. Finally, we use the results from above to define a new approximation.

   (a) Based on the evidence from Problem 2, find the best combination of the trapezoid approximation and the midpoint approximation to produce a better approximation than either alone. If all has gone well, you should arrive at the approximation known as **Simpson's rule**.

   (b) Use Simpson's rule to approximate $\displaystyle\int_0^2 e^{-x^2}\,dx$ with $n = 5$ subintervals.